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SOME ASPECTS ON REPRESENTATION THEORY OF INVOLUTIVE GROUP ALGEBRAS

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ABSTRACT

Let A be a separable C*-algebra and G_1 and G_2 be two locally compact groups. We consider two C*-dynamical systems (A, G_1, θ) and (A, G_2, ξ) with two states f and g which are stationary for the group morphisms θ and ξ respectively. Let $H_{\pi f}$ and $H_{\pi g}$ be the associated Hilbert spaces and $L^1(G_1, A; \theta)$ and $L^1(G_2, A; \xi)$ be the involutive group algebras. For every element u in $H_{\pi f} \otimes_\gamma H_{\pi g}$, we obtain a positive form τ on the projective tensor product of $L^1(G_1, A; \theta)$ and $L^1(G_2, A; \xi)$. Again, for two specific left ideals N_{1j} and N_{2j} of $L^1(G_1, A; \theta)$ and $L^1(G_2, A; \xi)$ respectively, we show that $\cap(N_{1j} \otimes N_{2j})+\cap(N_{1j}^* \otimes N_{2j}^*)$ is contained in ker τ.

KEY WORDS: Involutive group algebra/ tensor product / C*-dynamical system / positive form

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1. INTRODUCTION

In [4], Dixmier (1982) presented an interesting discussion regarding positive form and representation theory of involutive algebras. In this paper, we construct a certain type of positive form for the projective tensor product of involutive group algebras.

Before going to prove our main results, we present some basic terminologies (refer to [1],[2],[4],[6],[7]) which are useful for our discussion.

Definition 1.1. A C*-dynamical system is a triple (A, G, θ) where A is a C*-algebra, G is a locally compact group and θ: $G \rightarrow \mathrm{Aut}(A)$ is a group morphism which is continuous in the sense that for each $z \in A$, the function $x \rightarrow \theta(x)z$ is a continuous function from G to A (refer to [1]).

Definition 1.2. Let A be a separable C*-algebra and G be a locally compact group. Let K(G, A) be the collection of all functions $f : G \rightarrow A$ with compact support. Taking $\|f\|_1 = \int_G \|f(s)\|\, ds < \infty$, where "$ds$" denotes the Haar measure on G, we have, the generalized group algebra $L^1(G, A)$ is the completion of K(G, A) in this norm. The multiplication on $L^1(G, A)$ is defined by: $(f.g)(t) = \int_G f(s)\theta(s)(g(s^{-1}t))ds$, as a sort of twisted convolution, and also an involution is defined by $f^*(t) = \Delta(t)^{-1}\theta(t)(\,f(t^{-1})^*)$, where $f, g \in$ K(G, A) and $s, t \in G$ (refer to [1]). Under these operations, $L^1(G, A)$ becomes an involutive Banach algebra and it is denoted by $L^1(G, A; \theta)$.

Definition 1.3. Let X and Y be two normed spaces. Then the projective tensor norm $\|\,.\,\|_\gamma$ on $X \otimes Y$ is defined as: $\|\, u \,\|_\gamma = \inf \{ \sum_i \|x_i\|.\|y_i\| : u = \sum_i x_i \otimes y_i \}$, where the infimum is taken over all (finite) representations of u. The completion of ($X \otimes Y$, $\|\,.\,\|_\gamma$) is called the projective tensor product of X and Y, and is denoted by $X \otimes_\gamma Y$ (refer to [2]).

Definition 1.4. Let A be an involutive algebra and $\beta(H)$ be the collection of all bounded linear operators on a Hilbert space H. A mapping θ of A into $\beta(H)$ is called a representation of A in H if $\theta(x+y)=\theta(x)+\theta(y)$, $\theta(\lambda x)=\lambda\theta(x)$, $\theta(xy)=\theta(x)\theta(y)$, $\theta(x^*)=\theta(x)^*$, for $x, y\in A$, $\lambda\in$ C (refer to [4]).

Definition 1.5. A linear form f on an involutive algebra A is said to be positive if $f(x^*x)\geq 0$ for each $x\in A$. If A is a normed involutive algebra, a continuous positive linear form f on A is called a state if $\|f\|=1$ (refer to [4]).

Definition 1.6. Let (A, G, θ) be a C*-dynamical system. A covariant representation (refer to [1], [5]) of the system (A, G, θ) is a triple (π, u, H), where (π, H) is a representation of A; (u, H) is a unitary representation of G and $\pi((\theta(s))(z)) = u(s)\ \pi(z)\ u(s)^*$, for all z in A and s in G. Every C*-dynamical system has a covariant representation (refer to [1]).

Definition 1.7. Let A be an involutive algebra and θ be a representation of A in the Hilbert space H. Let $p\in H$ and S be the closed subspace of H generated by $\theta(x)p$, $x\in A$. If $S=H$, then θ is called a non-degenerate representation (refer to [4]).

2. MAIN RESULTS

Now, we consider the C*-dynamical system (A, G, θ), where A is a separable C*-algebra. Relating covariant representation of (A, G, θ) with the non-degenerate representations of the involutive algebra $L^1(G, A; \theta)$, Pedersen obtained the following result.

Lemma 2.1 [1] If (π, u, H) is a covariant representation of (A, G, θ), then there is a non-degenerate representation $(\pi\times u, H)$ of $L^1(G, A; \theta)$ such that

$$(\pi\times u)(f) = \int_G \pi(f(s))u(s)ds \quad \text{for every } f\in \mathrm{K}(G, A).$$

Also, the correspondence $(\pi, u, H) \rightarrow (\pi\times u, H)$ is a bijection onto the set of non-degenerate representations of $L^1(G, A; \theta)$.

For two given C*-dynamical systems (A, G_1, θ) and (A, G_2, ξ), we now derive a positive form for the projective tensor product $L^1(G_1, A; \theta)\otimes_\gamma L^1(G_2, A; \xi)$.

Theorem 2.2 Let (A, G_1, θ) and (A, G_2, ξ) be two given C*-dynamical systems, where A is a separable C*-algebra and let $x\rightarrow\theta(x)$ is a representation of G_1, $y\rightarrow\xi(y)$ is a representation of G_2. Let f and g be two states of A which are stationary for the group morphisms θ and ξ respectively, and let $H_{\pi f}$ and $H_{\pi g}$ be the associated Hilbert spaces. Then for every element $u \in H_{\pi f} \otimes_\gamma H_{\pi g}$, there is a positive form τ on $L^1(G_1, A; \theta) \otimes_\gamma L^1(G_2, A; \xi)$ with

$$\|\tau\| = \sum_j \|p_j\|^2.\|q_j\|^2 \text{ , where } u = \sum_j p_j \otimes q_j \text{ .}$$

Proof. For the C*-dynamical system (A, G_1, θ), we have, the group morphism θ: $G_1\rightarrow\text{Aut}(A)$ is continuous in the sense that for each $z \in A$, the function $x\rightarrow\theta(x)z$ is a continuous function from G_1 to the C*-algebra A. Again the state f on A is stationary for θ, i.e., $f(\theta(x)z) = f(z)$, for each $z \in A$ and $x \in G_1$. Let π_f be the representation defined by f and $H_{\pi f}$ be the associated Hilbert space. Also given that $x\rightarrow\theta(x)$ is a representation of G_1. Under these conditions, there exists a unique continuous unitary representation ρ_1 of G_1 (refer to [4]) in $H_{\pi f}$, satisfying $\pi_f(\theta(x)z) = \rho_1(x)\, \pi_f(z)\, \rho_1(x^{-1}) = \rho_1(x)\, \pi_f(z)\, \rho_1(x)^*$, (since ρ_1 is unitary) for each $z \in A$, $x \in G_1$.

Thus (π_f, ρ_1, θ) is a covariant representation for the system (A, G_1, θ). By Lemma 2.1, there is a non-degenerate representation $(\pi_f \times \rho_1, H_{\pi f})$ of $L^1(G_1, A; \theta)$ such that

$$(\pi_f \times \rho_1)(x) = \int_G \pi_f(x(s))\rho_1(s)ds \quad \text{for every } x \in K(G_1, A).$$

Similarly, for the continuous unitary representation ρ_2 of G_2 in $H_{\pi g}$, we get a non-degenerate representation $(\pi_g \times \rho_2, H_{\pi g})$ of $L^1(G_2, A; \xi)$.

We denote the representations $\pi_f \times \rho_1$ and $\pi_g \times \rho_2$ by π_1 and π_2 respectively.

Now, for $u = \sum_j p_j \otimes q_j \in H_{\pi f} \otimes_\gamma H_{\pi g}$, we define τ on $L^1(G_1, A; \theta) \otimes_\gamma L^1(G_2, A; \xi)$ by $\tau(\sum_i x_i \otimes y_i) = \sum_{i,j} \left\langle \pi_1(x_i)p_j, p_j \right\rangle \left\langle \pi_2(y_i)q_j, q_j \right\rangle$.

We know that for each p_j, the representation π_1 defines a positive form $f_{1j}: x \to \left\langle \pi_1(x)p_j,\ p_j \right\rangle$ on $L^1(G_1, A; \theta)$.

Similarly, for each q_j, we have, f_{2j}: $y \to \left\langle \pi_2(y)q_j,\ q_j \right\rangle$ is a positive form on $L^1(G_2, A; \xi)$. Thus, $\tau(\sum_i x_i \otimes y_i) = \sum_{i,j} f_{1j}(x_i) f_{2j}(y_i)$.

Then τ is a positive form on $L^1(G_1, A; \theta) \otimes_\gamma L^1(G_2, A; \xi)$.

Now, $\left\| \tau(\sum_i x_i \otimes y_i) \right\| = \left\| \sum_{i,j} f_{1j}(x_i) f_{2j}(y_i) \right\| \le \sum_j \| f_{1j} \| . \| f_{2j} \| . \sum_i \|x_i\| \|y_i\|$.

Again, $\| f_{1j}(x) \| \le \|\pi_1(x)\| . \|p_j\|^2 \le \|x\| . \|p_j\|^2$ and so, $\| f_{1j} \| \le \|p_j\|^2$.

Similarly, $\| f_{2j} \| \le \|q_j\|^2$.

Then $\|\tau\| \le \sum_j \| f_{1j} \| . \| f_{2j} \| \le \sum_j \|p_j\|^2 . \|q_j\|^2$.

By [4], we have, the C*-algebra A has an increasing approximate identity bounded by 1. So, $L^1(G_1, A; \theta)$ has an approximate identity, say, $\{e_i\}_i$ bounded by 1.

Similarly, let $\{u_k\}_k$ be the approximate identity in $L^1(G_2, A; \xi)$. So, $\{e_i \otimes u_k\}_{i,k}$ is an approximate identity for $L^1(G_1, A; \theta) \otimes_\gamma L^1(G_2, A; \xi)$.

By [4], $\|\tau\| = \lim\limits_{i,k} \tau(e_i \otimes u_k)$

$$= \lim_{i,k} \sum_j \langle \pi_1(e_i) p_j, p_j \rangle \langle \pi_2(u_k) q_j, q_j \rangle$$

$$= \sum_j \lim_i \langle \pi_1(e_i) p_j, p_j \rangle \lim_k \langle \pi_2(u_k) q_j, q_j \rangle$$

$$= \sum_j \langle p_j, p_j \rangle \langle q_j, q_j \rangle,$$ [$\{e_i\}_i$ and $\{u_k\}_k$ being approximate identities, and π_1, π_2 being non-degenerate representations, both $\pi_1(e_i)$ and $\pi_2(u_k)$ tends strongly to I]

$$= \sum_j \|p_j\|^2 . \|q_j\|^2$$

Deduction 2.3 For each of the positive forms $f_{1\,j}$ and $f_{2\,j}$ defined in the Theorem 2.2, let N_{1j} and $N_{2\,j}$ denote the left ideals: $N_{1j} = \{x \in L^1(G_1, A; \theta): f_{1\,j}(x^*x) = 0\}$ and $N_{2\,j} = \{y \in L^1(G_2, A; \xi): f_{2\,j}(y^*y) = 0\}$. Then $\cap(N_{1j} \otimes N_{2j}) + \cap(N_{1j}^* \otimes N_{2j}^*) \subseteq \ker \tau$.

Proof. For $\sum_i x_i \otimes y_i \in L^1(G_1, A; \theta) \otimes_\gamma L^1(G_2, A; \xi)$, we have,

$$\left|\tau(\sum_i x_i \otimes y_i)\right| = \left|\sum_{i,j} f_{1j}(x_i) f_{2j}(y_i)\right| \leq \sum_{i,j} \left|f_{1j}(x_i)\right| . \left|f_{2j}(y_i)\right|$$

$$\leq \left(\sum_{i,j} \left|f_{1j}(x_i)\right|^2\right)^{1/2} . \left(\sum_{i,j} \left|f_{2j}(y_i)\right|^2\right)^{1/2}$$

$$\Rightarrow \left|\tau(\sum_i x_i \otimes y_i)\right|^2 \leq \left(\sum_{i,j} \left|f_{1j}(x_i)\right|^2\right) . \left(\sum_{i,j} \left|f_{2j}(y_i)\right|^2\right)$$

$$\leq \left(\sum_{i,j} \left\|f_{1j}\right\| f_{1j}(x_i * x_i)\right) . \left(\sum_{i,j} \left\|f_{2j}\right\| f_{2j}(y_i * y_i)\right) \qquad (6.6)$$

If $u = \sum_i x_i \otimes y_i \in \cap(N_{1j} \otimes N_{2j})$, then $x_i \otimes y_i \in N_{1j} \otimes N_{2j}$ for each i, and for each j. So, $f_{1j}(x_i{*}x_i) = 0$ and $f_{2j}(y_i{*}y_i) = 0$ for each i, and for each j. By (6.6), $u \in \ker \tau$, and thus, $\cap(N_{1j} \otimes N_{2j}) \subseteq \ker \tau$. Again, $\cap(N_{1j}{*} \otimes N_{2j}{*}) \subseteq (\ker \tau)^* = \ker \tau$.

Thus, $\cap(N_{1j} \otimes N_{2j}) + \cap(N_{1j}{*} \otimes N_{2j}{*}) \subseteq \ker \tau$.

3. REMARK

Given a C*-dynamical system (A, G, θ), the crossed product of A by the action θ of G, denoted by $A \times_\theta G$, is the enveloping C*-algebra of $L^1(G, A; \theta)$ (refer to [1]). Regarding representations of an involutive Banach algebra X and its enveloping C*-algebra Y, we have,

Lemma 3.1 [4] Let X be an involutive Banach algebra having an approximate identity, Y the enveloping C*-algebra of X and ι the canonical map of X into Y. Then

(i) If π is a representation of X, there is exactly one representation ρ of Y such that $\pi = \rho o \iota$, and $\rho(Y)$ is the C*-algebra generated by $\pi(X)$.

(ii) The map $\pi \rightarrow \rho$ is a bijection of the set of representations of X onto the set of representations of Y.

(iii) ρ is non-degenerate if and only if π is non-degenerate.

Now, from Theorem 2.2, corresponding to the non-degenerate representations π_1 and π_2 of $L^1(G_1, A; \theta)$ and $L^1(G_2, A; \xi)$, using above Lemma, we get the non-degenerate representations ρ_1 and ρ_2 of $A \times_\theta G_1$ and $A \times_\xi G_2$ respectively. So, analogous results as in Theorem 2.2 can be obtained in case of $(A \times_\theta G_1) \otimes_\gamma (A \times_\xi G_2)$.

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