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RAYLEIGH-TAYLOR INSTABILITY OF COUPLE STRESS FLUID THROUGH POROUS MEDIA IN A FLUID LAYER OF FINITE THICKNESS

Mr.M. Siddalinga Prasad
Department of Mathematics, Siddaganga Institute of Technology, Tumkur – 572 103, Karnataka, India, msp_maths@rediffmail.com

ABSTRACT

Linear Rayleigh-Taylor instability of couple stress fluid layer of finite thickness through porous media has been studied in the creeping flow limit using normal modes. It is shown that the porous parameter, couple stress parameter have reducing effect on the growth rate whereas layer thickness has increasing effect on the growth rate. Shape of the dispersion curve is controlled only by the ratio of surface tension to pressure gradient.

KEY WORDS: Growth rate; Couple stress parameter; Porous parameter; Dispersion relation

1. INTRODUCTION

Torricelli in the 17^{th} century showed that the atmospheric pressure was able to produce a force equivalent to the weight of a 10-m-long column of water, Batchelor (1973). But, from our every day experience we know that the atmospheric pressure cannot support even a few centimeters of water with in a container when it is inverted. The reason for this apparent contradiction is the characteristic of instability of the equilibrium configuration in which water, the denser fluid lies above the less dense air, Sharp (1984) and Benjamin (1999). The instability of dense fluid above a less dense fluid in the gravitational field is known as the Rayleigh-Taylor Instability (RTI) in honor of Lord Rayleigh

(1900), who first described it mathematically and Sir G. Taylor (1950), who demonstrated that the instability can also occur in accelerated fluids.

RTI plays an important role in many natural processes ranging from coastal upwelling, which helps to renew the nutrients near the surface of the sea, Cui and Street (2004), to the ignition of a supernova at the end of life of some stars, Bychkoov and Liberman (1995). RTI also exists in the formation of some astrophysical structures such as the supernova remnants in the Eagle and Crab nebula, Ribeyre et. al., (2004), in the air bubble formation of the blood of deep-sea divers, Racca and Annett (1985). It is also important in plasma physics and is one of the main concerns in inertial fusion, Piriz, et. al., (1997).

RTI of incompressible viscous fluids of different densities has been studied by Chandrasekhar (1961). Babchin et. al. (1983) have studied the non-linear saturation of RTI in a thin Newtonian fluid film when the wave length is much greater than the film thickness. Later Brown (1989) relaxed the assumption on the wavelength and studied the RTI in a finite thin layer of a viscous fluid using the combined Stokes and lubrication approximation as in Babchin et. al. (1983). Brown (1989) has shown that RTI of fluid layer of finite thickness, if stagnant would rupture as a consequence of the instability. The growth of such instability often leads to undesirable effects like mixing and/or shell breaking when the heavier fluid is in the form of a shell of finite thickness. In situation like these, it is important and desirable to find a mechanism to suppress the growth rate of RTI and is the aim of the present paper.

In literature, the different types of density gradients, the shear rates and surface tension (see Kull (1991)) have been used to suppress the growth rate of RTI. Later, Rudraiah et. al. (1996, 1997,1998,2003,2004,2007) have studied the effects of viscosity stratification, oblique magnetic field, porous lining with Beavers and Joseph (1967) slip condition, porous lining with Saffman (1971) slip condition, magnetic field and electric field on RTI to reduce the growth rate.

All the works cited above are concerned with only the Newtonian fluids. With the growing importance of non-Newtonian fluids in modern technology, the investigation of such fluids is desirable. Stokes (1966) has formulated the theory of couple-stress fluid. The presence of small amount of additives in a lubricant can improve the bearing performance by increasing the lubricant viscosity and thus producing an increase in the load capacity. These additives in a lubricant reduce the coefficient of friction and also increase the temperature range in which the bearing can operate.

A number of theories of the microcontinuum has been proposed and applied by Stokes (1966), Lai et. al. (1978), and Walicka (1994). The theory of Stokes (1966) allows for polar effects such as the presence of couple-stresses, body couples and has been applied to the study of some simple lubrication problems by Sinha et. al. (1981), Bujurke and Jayaraman (1982), Lin (1996), and Naduvinamani et. al. (2005). In all the above studies, the medium has been considered to be non-porous.

In recent years, the investigation of the flow of fluids through porous media has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. A great number of applications in geophysics are cited in the books by Phillips (1991), Nield and Bejan (1999). When the fluid permeates a porous material, the gross effect is represented by the Darcy's law. As a result of this macroscopic law, the usual viscous and couple-stress viscous terms in the equation of couple-stress fluid motion are replaced by the resistance term $-\frac{1}{k_1}\left(\mu - \mu'\nabla^2\right)\vec{q}$ where μ and μ' are viscosity and couple-stress viscosity, k_1 permeability of the porous medium, $\vec{q}$ is Darcian (filter) velocity.

Keeping in mind the importance and applications of non-Newtonian fluids and porous medium; the present paper attempts to study the RTI of couple stress fluid layer of finite thickness through porous media.

To achieve this objective, the plan of the paper is as follows. In section 2, on mathematical formulation of the problem, the physical configuration, the basic equations and the required boundary conditions are given. In section 3 dispersion relation is derived. In final section conclusions are drawn using the dispersion relation and the results are depicted graphically.

2. MATHEMATICAL FORMULATION

We consider the instability of one interface of an incompressible, couple stress fluid layer of finite thickness flowing through an isotropic and homogeneous porous medium as shown in Fig. 1. The relevant parameters are h layer thickness, γ surface tension and δ the stress gradient.

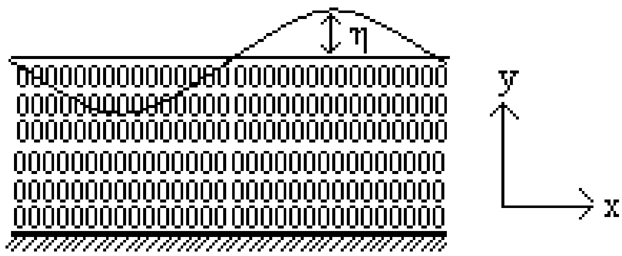

Figure 1 The geometry of the perturbation

The basic equations governing the motion of a couple-stress fluid through porous medium are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}(\vec{q}\cdot\nabla)\vec{q}\right]=-\nabla p+\rho\vec{g}-\frac{1}{k_1}\left(\mu-\mu'\nabla^2\right)\vec{q}\,, \tag{1}$$

$$\nabla\cdot\vec{q}=0\,, \tag{2}$$

where p is pressure, ρ density of the couple stress fluid and ε is the porosity of the porous medium.

Equations (1) and (2) represent the equation of motion and continuity respectively for the couple stress fluid. The initial state is stationary. The characteristic of the equilibrium of this static state is determined by imposing small perturbations and following its further evolution. The linearized perturbation equations governing the motion and following the assumptions of Brown (1989) are

$$\frac{\partial p}{\partial x} = -\frac{\mu}{k_1} u + \frac{\mu'}{k_1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{3}$$

$$\frac{\partial p}{\partial y} = -\frac{\mu}{k_1} v + \frac{\mu'}{k_1}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

where u and v are the components of velocity along x- and y- directions and it has been assumed that there is no motion in the z-direction.

The relevant boundary conditions are, no slip at the rigid boundary

$$u = v = 0 \quad at \quad y = 0, \tag{6}$$

and vanishing of tangential stress at the interface

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad at \quad y = h. \tag{7}$$

The continuity of normal stress at the interface leads to

$$p = -\delta\eta - \gamma\frac{\partial^2 \eta}{\partial x^2} + \left(\mu - \frac{\mu'}{h^2}\right)\frac{\partial v}{\partial y} \qquad at \qquad y = h. \tag{8}$$

Here $\frac{\partial^2 \eta}{\partial x^2}$ is the curvature of the interface and η is the elevation of the interface, satisfying the dynamic boundary condition

$$v = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \qquad at \qquad y = h\,, \tag{9}$$

for linear theory, we ignore the second term on right hand side of Eq.(9).

3. DISPERSION RELATION

Analyzing the perturbations into normal modes, we assume

$$[u, v, \eta] = [\hat{u}(y), \hat{v}(y), \hat{\eta}]e^{ikx+nt}\,, \tag{10}$$

where k and n represent wave number and frequency of perturbation.

Eliminating p from Eqs. (3) and (4) and using Eq. (10),

$$\left[-\mu i k + \mu'\left(-i k^3 + i k D^2\right)\right]\hat{v} = \left[-\mu D + \mu'\left(-k^2 D + D^3\right)\right]\hat{u}\,. \tag{11}$$

Equations (5) and (11) on using Eq. (10), are

$$i k \hat{u} + D\hat{v} = 0\,, \tag{12}$$

$$\left[D^4 - \left(2k^2 + \alpha\right)D^2 + k^2\left(k^2 + \alpha\right)\right]\hat{v} = 0\,, \tag{13}$$

where $D = \frac{d}{dy}$ and $\alpha = \frac{\mu}{\mu'}$ a constant with the dimension of length square.

Equation (13) reduces to the result of Brown (1989) when $\alpha = 0$.

Equation (13) satisfies,

$$D\hat{v} = \hat{v} = 0 \quad at \quad y = 0\,, \tag{14}$$

and

$$\left(D^2+k^2\right)\hat{v}=0 \quad at \quad y=h. \tag{15}$$

From Eqs. (9), (3) and (8) on using Eq. (10), we get

$$n=\frac{k^2\left(\delta-\gamma k^2\right)\hat{v}(h)}{\mu'\left\{\left[\frac{\alpha+k^2}{k_1}+k^2\left(\alpha-\frac{1}{h^2}\right)\right]D\hat{v}-\frac{D^3\hat{v}}{k_1}\right\}}. \tag{16}$$

Solution of Eq.(13) when $\alpha \neq 0$ is

$$\hat{v}=A_1\,Sh(ky)+B_1\,Ch(ky)+C_1Sh\left(\sqrt{k^2+\alpha}\;y\right)+D_1Ch\left(\sqrt{k^2+\alpha}\;y\right) \tag{17}$$

where A_1, B_1, C_1, D_1 are constants of integration and are obtained using Eqs. (14) and (15),

$$B_1=-D_1, \qquad A_1=-\frac{\sqrt{k^2+\alpha}}{k}C_1, \qquad C_1=-\frac{a_0}{a_1}D_1,$$

$$a_0=\left(2k^2+\alpha\right)\dot{S}_1-2k\sqrt{k^2+\alpha}\,\dot{S},$$

$$a_1=\left(2k^2+\alpha\right)\dot{C}_1-2k^2\,\dot{C}$$

$$\dot{S}=Sinh(k\,h), \quad \dot{C}=Cosh(k\,h), \quad \dot{S}_1=Sinh\left(h\sqrt{k^2+\alpha}\right),$$

$$\dot{C}_1=Cosh\left(h\sqrt{k^2+\alpha}\right).$$

The dispersion relation Eq. (16) on using Eq. (17) takes the form,

$$n=\frac{k^2\left(\delta-\gamma k^2\right)}{\mu'}N, \tag{18}$$

where

$$N = \frac{a_1\left(\frac{\sqrt{k^2+\alpha}}{k}\dot{S}-\dot{S}_1\right)+a_0\left(\dot{C}_1-\dot{C}\right)}{a_1 T_1 + a_0 T_0},$$

$$T_1 = \sqrt{k^2+\alpha}\left\{\left[\frac{\alpha+k^2}{k_1}+k^2\left(\alpha-\frac{1}{h^2}\right)\right]\left(\dot{C}-\dot{C}_1\right)+\frac{-k^2\dot{C}+\left(k^2+\alpha\right)\dot{C}_1}{k_1}\right\},$$

$$T_0 = \left\{\left[\frac{\alpha+k^2}{k_1}+k^2\left(\alpha-\frac{1}{h^2}\right)\right]\left(\sqrt{k^2+\alpha}\,\dot{S}_1-k\,\dot{S}\right)+\frac{-\left(k^2+\alpha\right)^{\frac{3}{2}}\dot{S}_1+k^3\dot{S}}{k_1}\right\}.$$

4. DISCUSSION

Dispersion relation, Eq. (18) clearly shows that the stability of the system is controlled by $\left(\frac{\delta}{\gamma}\right)$ which is similar to the case of Brown (1989) with permeability of porous layer, couple stress α and layer thickness affecting the rate of growth of instability. Note that in the case of Brown (1989) only layer thickness affects the rate of growth of instability where as in the present case there are two more factors in addition to layer thickness affecting rate of growth of instability.

Equation (18) is made dimensionless using scales $\frac{\sqrt{\delta\gamma}}{\mu}$ for frequency, $\frac{1}{\lambda}$ for wave number, $\lambda = \sqrt{\frac{\gamma}{\delta}}$ for layer thickness. The graph of frequency (*n*) as a function of wave number (k) are plotted for different parameters, namely $\hat{h} = \frac{h}{\lambda}$ the layer thickness, $\sigma = \frac{\sqrt{k_1}}{\lambda}$ the porous parameter and $f = \frac{\mu'}{\mu\lambda^2}$ the couple stress parameter to know whether σ and f augment or suppress the growth rate of instability as conclusions cannot be drawn from Eq.(18) because of the complicated structure of N. In each case there are two sets of graphs one for small value of f and the other for large value of f, couple stress parameter.

Figures (2a), (2b), (5a) and (5b) are the graphs of wave number versus frequency for different values of $\hat{h}$ with σ and f fixed. From these figures it is clear that as the layer thickness is increased towards λ, it starts to affect the dispersion curve indicating that it enters a regime where the wavelength of the maximum growth rate is of the order of layer thickness and hence the instability grows with $\hat{h}$.

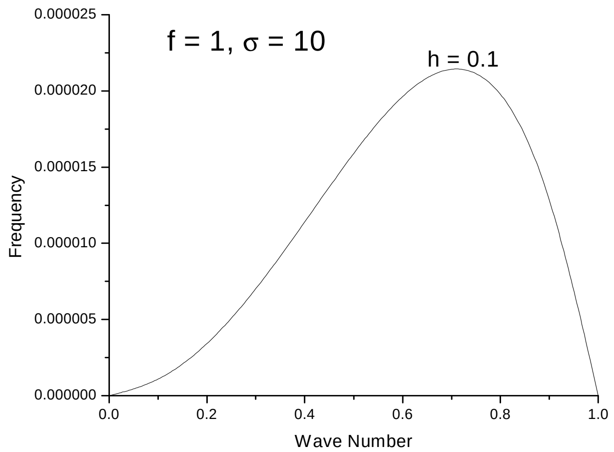

Figure 2 (a) : Plots of growth rate as a function of wave number for small *f* and $\hat{h}$

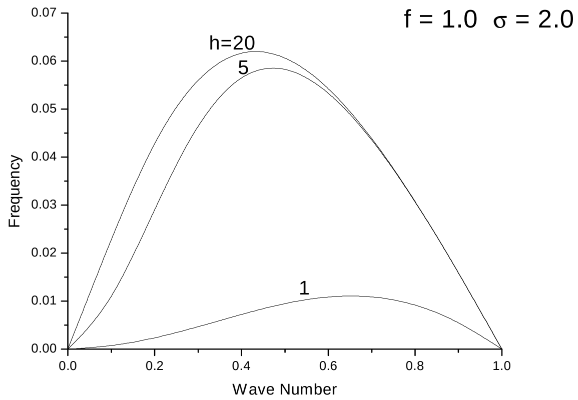

Figure 2(b) : Plots of growth rate as a function of wave number for small *f* by varying $\hat{h}$

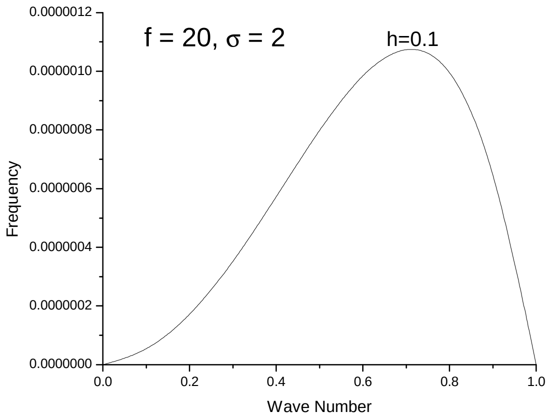

Figure 5(a): Plots of growth rate as a function of wave number for large *f* and small $\hat{h}$

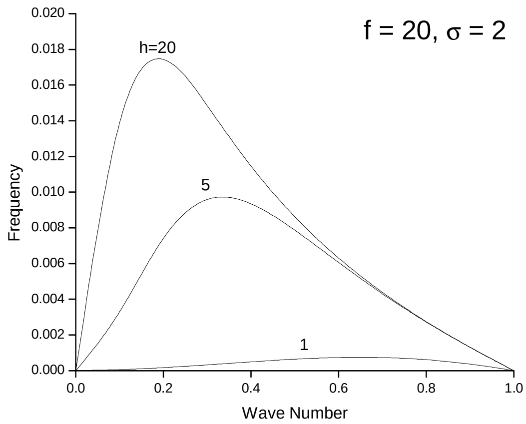

Figure 5(b): Plots of growth rate as a function of wave number for large *f* by varying $\hat{h}$

Figures (3) and (6) are the graphs of wave number versus frequency for different values of σ with *f* and $\hat{h}$ fixed. As the porous parameter is increased the growth rate of instability reduces considerably, this is because of dampening effect of Darcy resistance. Therefore, the effect of porous parameter is to stabilize Rayleigh Taylor modes, which was otherwise unstable in its absence.

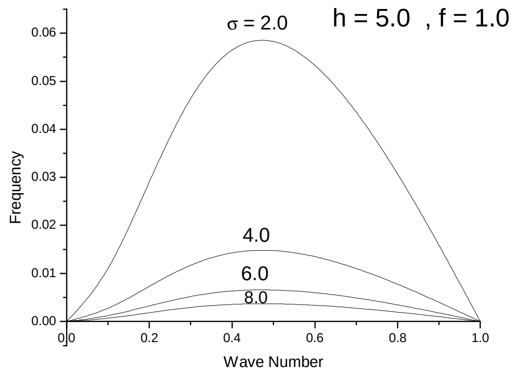

Figure 3 : Plots of growth rate as a function of wave number for small *f* by varying σ

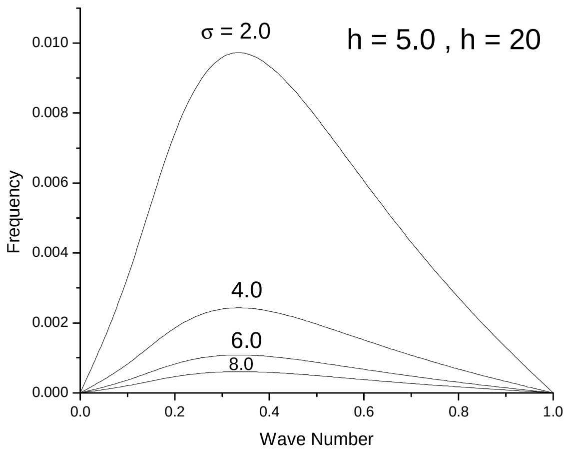

Figure 6 : Plots of growth rate as a function of wave number for large *f* by varying σ

he effect of couple stress parameter is shown in Figures (4) and (7), it is clear that increase in f would decrease the growth rate. Note that the wavelength of maximum growth rate is a function of couples stress parameter up to the ratio of $\left(\frac{f}{\sigma}\right)$ is of the order of $\hat{h}$ and beyond that it enters a region where the wavelength of the maximum growth rate become constant as depicted in Figures (4) and (7).

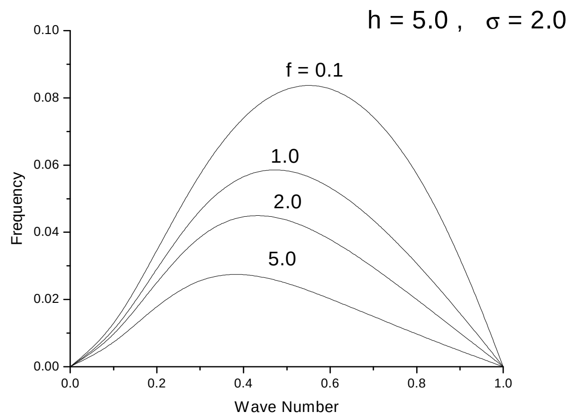

Figure 4: Plots of growth rate as a function of wave number by varying *f*

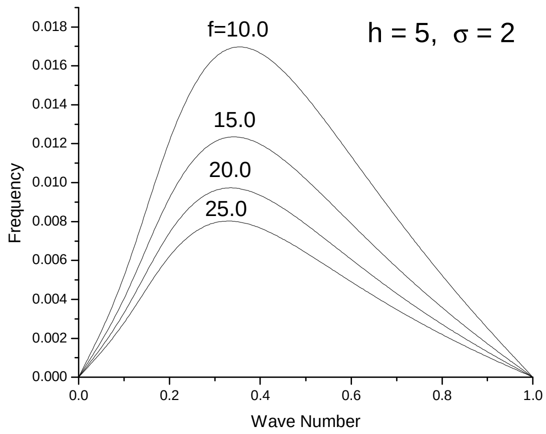

Figure 7 : Plots of growth rate as a function of wave number by varying *f*

Finally, we conclude that porous parameter and couple stress parameter have reducing effect on the growth rate of Rayleigh-Taylor Instability. Therefore, proper choice of structure of porous material and couple stress fluid it is possible to reduce the growth rate of Rayleigh-Taylor Instability as expected in many practical applications cited in the introduction.

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