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NUMERICAL METHOD FOR SOLVING SINGULAR NONLINEAR FIRST ORDER INITIAL VALUE PROBLEM

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ABSTRACT

In this paper, a reproducing kernel method is constructed in order to solve a class of non-linear singular first order initial value problem.

The exact solution of linear problem can be expressed in the form of series and approximate solution of nonlinear problem is given by iterative formula. The performance of the method is illustrated by two numerical examples.

Keywords: Gram-Schmidt orthogonal process; Reproducing kernel space.

1. INTRODUCTION

Consider the following nonlinear first order initial value problem:

$$\begin{cases} a(x)u^{(1)}(x)+u(x)=f(x,u), & 0<x\le 1 \\ u(0)=0 \end{cases} \tag{1.1}$$

where $a(x)$ is bounded function.

Singular boundary value problems for ordinary differential equation arise very frequently in several areas of science and engineering. Singular boundary value problems have been studied by several authors. Zhou and Wei [8] presented a two-step algorithm for solving singular linear systems. Abukhaled, Khuri and Sayfy [4] presented B-spline method for numerically solving singular two-point boundary value problems for certain ordinary differential equation having singular coefficients. Ghazala and Hamood [1] discussed computational method for solving singularly perturbed first order initial value problem with initial conditions at any end point. Sallam [6] constructed a global method, based on quintic C^1 spline, for the integration of first order ordinary initial value problems (IVPs) including stiff equations and those possessing oscillatory solutions as well. Venkatesulu and Srinivasu [7] solved nonstandard first order initial value problem. So far, many classical problems in probability and statistics, had been solved in reproducing kernel space [2, 3].

The paper is organized into four sections including the introduction. In section 2, construction of reproducing kernel is presented for solving (1.1). The solution of the problem in reproducing kernel Hilbert space is presented in section 3. In section 4, two numerical examples are given to demonstrate the accuracy of present method method.

2. REPRODUCING KERNEL SPACES

Reproducing kernel space $W_2^2[0,1]$

The inner product space $W_2^2[0,1]$ is defined by $W_2^2[0,1]=\{u(x)\big| u^{(i)}, i=0,1$ are absolutely continuous real valued functions in [0,1], $u, u^{(1)}u^{(2)} \in L^2[0,1]\}$. The inner product and norm in $W_2^2[0,1]$ are defined as

$$< u(y), v(y) >_{W_2^2} = u(0)v(0)+u(1)v(1)+\int_0^1 u^{(2)}(y)v^{(2)}(y)dy \tag{2.1}$$

$$\| u \| = \sqrt{< u, u >_{W_2^2}} \quad u(x), v(x) \in W_2^2[0,1] \tag{2.2}$$

Reproducing kernel space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1]=\{u(x)\,|\,u$ is absolutely continuous real valued functions in [0,1], $u,u^{(1)}u^{(2)}\in L^2[0,1]\}$.

The inner product and norm in $W_2^2[0,1]$ are defined as

$$<u(y),v(y)>_{W_2^1}=\int_0^1 u(y)v(y)dy+\int_0^1 u^{(1)}(y)v^{(1)}(y)dy \qquad (2.3)$$

$$\|u\|=\sqrt{<u,u>_{W_2^1}},\qquad u(x),v(x)\in W_2^1[0,1] \qquad (2.4)$$

In [5], it was proved that $W_2^1[0,1]$ is reproducing kernel Hilbert space.

Theorem 2.1

The space $W_2^2[0,1]$ is a reproducing kernel Hilbert space. That is $\forall u(y)\in W_2^2[0,1]$ and each fixed $x\in[0,1]$, $y\in[0,1]$, there exists $R_x(y)\in W_2^2[0,1]$ such that $<u(y),R_x(y)>=u(x)$ and $R_x(y)$ is called the reproducing kernel function of space $W_2^2[0,1]$. The reproducing kernel function $R_x(y)$ is given by

$$R_x(y)=\begin{cases} c_1+c_2y+c_3y^2+c_4y^3 & y\le x,\\ d_1+d_2y+d_3y^2+d_4y^3 & y>x.\end{cases}$$

Proof. Since $R_x(y)\in W_2^2[0,1]$, and from Eq. (2. 1), it can be written as

$$<u(y),R_x(y)>=u(0)R_x(0)+u(1)R_x(1)+\int_0^1 R_x^{(3)}(y)u^{(3)}(y)dy \qquad (2.5)$$

Several integration of (2.5), gives $R_x(0) = 0.$ (2.6)

Since, $u \in W_2^2[0,1],\ u(0) = 0.$ If

$$R_x(0) - R_x^{(3)}(0) = 0, R_x^{(2)}(1) = 0 \ \textit{and} \ R_x^{(2)}(1) = 0 \tag{2.7}$$

then Eq. (2. 5) implies that

$$< u(y), R_x(y) >= \int_0^1 u(y)\, R_x^{(4)}(y))dy$$

For all $x \in [0,1]$ if $R_x(y)$ also satisfies

$$R_x^{(4)}(y) = \delta(y - x) \tag{2.8}$$

Then

$$< u(y), R_x(y) >= u(x) \tag{2.9}$$

When $y \# x$ characteristic equation of Eq. (2. 8) is given by $\lambda^4 = 0$ then the characteristic values can be determined whose multiplicity is 4. The reproducing kernel $R_x(y)$ can be defined as

$$R_x(y) = \begin{cases} c_1 + c_2 y + c_3 y^2 + c_4 y^3 & y \le x, \\ d_1 + d_2 y + d_3 y^2 + d_4 y^3 & y > x. \end{cases} \tag{2.10}$$

and let $R_x(y)$ satisfies

$$R_x^{(k)}(x+0) = R_x^{(k)}(x-0),\ \ k = 0,1,2 \tag{2.11}$$

and integrating (2. 10) from $x-\varepsilon\ to\ x+\varepsilon$ with respect to y and $x \to 0$ use jump degree of $R^{(3)}{}_x(y)$ at $y = x$

$$R_x^{(3)}(x+0) - R_x^{(3)}(x-0) = 1. \tag{2.12}$$

The coefficients c_i and d_i (i = 1, 2, 3, 4) can be determined from Eqns. (2.6), (2.7), (2.11) and (2.12).

3. THE EXACT AND APPROXIMATE SOLUTION

The solution of Eq. (1.1) is given in the reproducing kernel Hilbert space $W_2^2[0,1]$ and the linear operator $L: W_2^2[0,1] \to W_2^1[0,1]$ is bounded and defined as

$$(Lu)(x) = a(x)u^{(1)} + u \tag{3.1}$$

and equivalent form of Eq. (3. 1) is

$$(Lu)(x) = \begin{cases} f(x,u) & 0 < x \le 1 \\ u(0) = 0. \end{cases} \tag{3.2}$$

For a fix countable dense subset $D = \{x_i\}_{i=1}^{\infty}$ of the domain [0, 1], let

$$\varphi_i(y) = Q_{x_i}(y), \quad i \in N \tag{3.3}$$

where $Q_{x_i}(y) \in W_2^1[0,1]$ is reproducing kernel of $W_2^1[0,1]$. Further assume that $\psi_i(x) = (L^*\varphi_i)(x)$, where L^* denotes the adjoint operator of L.

Theorem 3.1

$\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^2[0,1]$ and $\psi_i(x) = L_y R_x(y)\big|_{y=x_i}$.

Proof

For each fixed $u(x) \in W_2^2[0,1]$, Let $\langle u(x), \psi_i(x) \rangle = 0\,(i = 1, 2, \ldots)$, which implies

$$\langle u(x), (L^* \varphi_i)(x) \rangle = < (Lu)(x), Q_{x_i}(x) > = (Lu)(x_i) = 0 \tag{3.4}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in *[0, 1]*, *(Lu)(x) = 0*, which implies *u = 0* from the existence of L^{-1}. From Eq. (2. 11), it can be written as

$$\psi_i(x) = (L^* \varphi_i)(x)$$

$$= < L^* \varphi_i(y), R_x(y) >$$

$$= < \varphi_i(y), LR_x(y) >$$

$$= L_y R_x(y) \Big|_{y = x_i}$$

The subscript *y* by the operator *L* indicates that *L* applies to the function of *y*. To orthonormalize the sequence $\{\psi_i(x)\}_{i=1}^{\infty}$ in the reproducing kernel space Gram-Schmidt process can be used as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \qquad i = 1, 2, 3, \ldots \tag{3.5}$$

Hence $\forall u(y) \in W_2^2[0,1]$ can be expanded in terms of Fourier series about normal orthogonal system

$$u(x) = \sum_{i=1}^{\infty} < u(x), \bar{\psi}_i(x) > \bar{\psi}_i(x) \tag{3.6}$$

Theorem 3.2

If $\{x_i\}_{i=1}^{\infty}$ is dense in *[0, 1]* and the solution of Eq. *(3. 2)* is unique, for all $u(x) \in W_2^2[0,1]$ the series is convergent in the norm $\| \cdot \|_{W_2^2}$. If *u(x)* is exact solution then the solution of Eq. (3. 2) has the form

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k)) \overline{\psi}_i(x)$$

Proof:

Since $u(x) \in W_2^2[0,1]$ and can be expanded in the form of Fourier series about normal orthogonal system $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$u(x) = \sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$$

Since the space $W_2^2[0,1]$ is Hilbert space so the series $\sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$ is convergent in the norm of $\| \cdot \|_{W_2^2}$.

From Eqns. (3. 5) and (3. 6) it can be written as

$$u(x) = \sum_{i=1}^{\infty} < u(x), \overline{\psi}_i(x) > \overline{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < u(x), \psi_i(x) > \overline{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < u(x), L^* \varphi_k(x) > \overline{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < Lu(x), \varphi_k(x) > \overline{\psi}_i(x)$$

If *u(x)* is the exact solution of Eq. (3. 2) and *Lu = f(x,u)*, then

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < f(x,u), \varphi_k(x) > \overline{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x,u) \overline{\psi}_i(x).$$

The approximate solution of *u(x)* is given by

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_x)) \overline{\psi}_i(x) \tag{3.7}$$

Eq. (1.1) is nonlinear, then approximate solution of (1.1) can be obtained using the following iteration formula:

$$\begin{cases} any\ fixed\ u_0(x) \in W_2^2[0,1] \\ u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x) \end{cases}$$

where

$$A_1 = \beta_{11} f(x_1, u_0(x_1))$$

$$A_2 = \sum_{k=1}^{2} \beta_{1k} f(x_1, u_{k-1}(x_1))$$

$$\vdots \quad \vdots \quad \vdots$$

$$A_n = \sum_{k=1}^{n} \beta_{nk} f(x_k, u_{k-1}(x_k))$$

Theorem 3.3

For each $u(x) \in W_2^2[0,1]$ and $\{r_n\}$ is the error between the approximate solution $u_n(x)$ and exact solution $u(x)$. Let $r_n^2 = \left\| u(x) - u_n(x) \right\|^2$, then sequence $\{r_n\}$ is monotone decreasing and $r_n \to o \ (n \to \infty)$.

Proof.

Given

$$\begin{aligned} r_n^2 &= \left\| u(x) - u_n(x) \right\|^2 \\ &= \left\| \sum_{i=n+1}^{\infty} \langle u(x), \overline{\psi}_i(x) \rangle \overline{\psi}_i(x) \right\|^2 \\ &= \sum_{i=n+1}^{\infty} (\langle u(x), \overline{\psi}_i(x) \rangle)^2 \end{aligned}$$

Clearly $r_{n-1} \geq r_n . \{r_n\}$ is monotone decreasing and from Theorem 3.3, it is noted that Eq. (3. 7) is convergent in the norm of $\left\| . \right\|_{W_2^2}$. *i.e* $r_n \to o \ (n \to \infty)$.

4. NUMERICAL EXAMPLES

Numerical example is studied to demonstrate the accuracy of the present method using mathematica 5.2.

Example 4.1 Consider the initial value problem

$$\begin{cases} x(1-x)u^{(1)}(x)+u(x)u(x)=f(x,u(x)), \quad 0<x\le 1 \\ u(0)=0. \end{cases} \tag{4.1}$$

The exact solution is $u(x)=x^2\sin x$ .The result is summarized in Table 1.

Example 4.2 Consider the initial value problem

$$\begin{cases} xu^{(1)}(x)+u(x)\sin(u(x))=f(x,u(x)), \quad 0<x\le 1 \\ u(0)=0. \end{cases} \tag{4.2}$$

The exact solution is $u(x)=\cos x-e^x$. The result is summarized in Table 2.

CONCLUSIONS

In this paper, reproducing kernel method is employed to solve a class of singular initial value problems. It is evident from the numerical examples that the method proposed in this paper gives the accurate results. The numerical results are displayed to demonstrate the validity of this method.

Table 1. The numerical results when *(n=56)*

x	Absolute Error	Relative Error
0.1	2.10 E-05	2.06E-02
0.2	3.39E-05	4.25E-03
0.3	4.68E--05	1.75E-03
0.4	5.79E--05	9.29E-05
0.5	6.83E--05	5.69E-04
0.6	7.71E--05	3.79E-04
0.7	8.41E--05	2.66E-04
0.8	8.94E--05	1.94E-04
0.9	9.26E--05	1.46E-04
1.0	8.95E--05	1.06E-04

Table 2. The numerical results when *(n=56)*

x	Absolute Error	Relative Error
0.1	2.54E-04	2.30E-04
0.2	1.99E-04	8.26E-04
0.3	1.53E-04	3.89E-04
0.4	1.16E-04	2.04E-04
0.5	8.81 E-05	1.14E-04
0.6	6.69E-05	6.71E-05
0.7	5.20E-05	4.17E-05
0.8	4.23E-05	2.76E-05
0.9	3.70E-05	2.01E-05
1.0	3.37E-05	1.54E-05

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