SOME ISOMORPHISM RESULTS ON PRODUCT FUZZY GRAPHS

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ABSTRACT

We have discussed notion of ring sum of product fuzzy graphs. We further give Three Independent Theorems based on Ring Sum, Join and Isomorphism of product fuzzy graph which are fuzzy version of classical graph theory.

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INTRODUCTION

Fuzzy logic has developed into a large and deep subject. Zadeh [7] addresses the terminology and stresses that fuzzy graphs are a generalization of the calculi of crisp graphs. Several other formulation of fuzzy graph problems have appeared in the literature. Koczy also gives a taxonomy of fuzzy graphs.

In [10] Rosendfeld and Yeh and Beng independently developed the theory of fuzzy graph. A fuzzy graph is a pair $G: (\sigma, \mu)$, where $\sigma$ is a fuzzy subset of $V$ and $\mu$ is fuzzy relation on $V$ such that, $\mu (uv) \leq \sigma (u) \land \sigma (v)$ for all $u, v \in V$.

A graph isomorphism[1] search is important problem of graph theory. It consists to define of bijective correspondence existence which preserve adjacent relation between vertex sets of two graphs [13]. In a case of fuzzy graphs the notion of isomorphism is fuzzy.

Ramaswamy and [11] replaced ‘minimum’ in the definition of fuzzy graph by ‘product’ and call the resulting structure product fuzzy graph and proved several results which are analogous to fuzzy graphs. Further they discussed a necessary an sufficient condition for a product partial fuzzy sub graph to be the multiplication of two product partial fuzzy sub graphs.
We have discussed notion of ring sum of product fuzzy graphs. We further provided Three Independent Theorems based on Ring Sum, Join and Isomorphism of product fuzzy graph which are fuzzy version of isomorphism on classical graphs [12].

**PRELIMINARY**

**Lemma 2.1** [11] Let G be a graph whose vertex set is $V$, $\sigma$ be a fuzzy sub set of $V$ and $\mu$ be a fuzzy sub set of $V \times V$ then the pair $(\sigma, \mu)$ is called product fuzzy graph if

$$\mu(u, v) \leq \sigma(u) \times \sigma(v), \quad \forall u, v \in V$$

**Remark**: If $(\sigma, \mu)$ is a product fuzzy graph and since $\sigma(u)$ and $\sigma(v)$ are less than or equal to 1, it follows that

$$\mu(u, v) \leq \sigma(u) \times \sigma(v) \leq \sigma(u) \cap \sigma(v), \quad \forall u, v \in V$$

Hence $(\sigma, \mu)$ is a fuzzy graph thus every product fuzzy graph is a fuzzy graph.

**Remark**: If $(\sigma, \mu)$ is a product fuzzy sub graph of G whose vertex set is $V$, we will assume that $\sigma(v) \neq 0 \quad \forall v \in V$ and $\mu$ is symmetric.

**Example 2.1** Let $V = \{u, v, w\}$, $\sigma$ be the fuzzy subset of $V$ defined as $\sigma(u) = \frac{1}{4}$, $\sigma(v) = \frac{1}{2}$ and $\sigma(w) = \frac{5}{6}$. Let $\mu$ be the fuzzy subset of $V \times V$ defined as $\mu(u, v) = \frac{1}{10}$, $\mu(v, w) = \frac{3}{8}$ and $\mu(u, w) = \frac{3}{16}$. It is easy to see that $(\sigma, \mu)$ is a product fuzzy graph and hence a fuzzy graph.
Remark:- Every product fuzzy graph is fuzzy graph but converse is not true.

**Lemma 2.2** [11] A product fuzzy graph \((\sigma, \mu)\) is said to be complete if

\[
\mu(u, v) = \sigma(u) \times \sigma(v), \forall u, v \in V
\]

**Lemma 2.3** The complement of a product fuzzy graph \((\sigma, \mu)\) is \((\overline{\sigma}, \overline{\mu})\) where

\[
\overline{\sigma} = \sigma \quad \text{and} \quad \overline{\mu}(u, v) = \sigma(u) \times \sigma(v) - \mu(u, v)
\]

It follows that \((\overline{\sigma}, \overline{\mu})\) itself is a product fuzzy graph. Also

\[
\overline{\mu}(u, v) = \sigma(u) \times \sigma(v) - \overline{\mu}(u, v)
\]

\[
= \sigma(u) \times \sigma(v) - [\sigma(u) \times \sigma(v) - \mu(u, v)]
\]

\[
= \mu(u, v) \forall u, v \in V
\]
Lemma 2.4 Consider the product fuzzy graph \( G_1^* (G_2, \mu_1) \) and \( G_2^* (G_2, \mu_2) \) with \( G_1^* = V_1 \) and \( G_2^* = V_2 \) are isomorphism between two product fuzzy graphs \( G_1 \) and \( G_2 \) is a bijective mapping

\[ h: V_1 \rightarrow V_2 \sigma_2(h(u)) = \sigma_2(h(u)) \quad \forall u \in V \quad \text{and} \quad \forall h(u) \in V_1 \]

and

\[ \mu_1(u,v) = \mu_2(h(u), h(v)) \quad \forall u,v \in V_1 \] we write in this case \( G_1 \cong G_2 \).

Remark 1- An automorphism of \( G \) is an isomorphism of \( G \) with itself.

Proposition 2.1 [14] Let \( G_2^* (V_2, E_2) \) be the intersection of graph \( G_1^* (V_1, E_2) \) and \( G_2: (V_2, E_2) \). Let \( G_1^* (G_2, \mu_1) \) and \( G_2: (G_2, \mu_2) \) are the product fuzzy graph of \( G_1 \& G_2 \) resp. then \((G_1 \cap G_2, \mu_1 \cap \mu_2)\)

is a product fuzzy graph of \( G \).

Lemma 2.5 [14] Let \( G_2: (V_2, E_2) \) and \( G_2: (V_2, E_2) \) with \( G_1: (V_1, E_2) \) and \( G_2: (V_2, E_2) \) and let \( G = G_1 \cup G_2: (V_1 \cup V_2, E_2 \cup E_2) \) be the union of \( G_1 \) and \( G_2 \). Then the union of two product fuzzy graphs \( G_1 \) and \( G_2 \) is defined as \( G = G_1 \cup G_2: (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2) \) where

\[
(\sigma_1 \cup \sigma_2)u =
\begin{cases} 
\sigma_1(u) & \text{if } u \in V_1 - V_2 \\
\sigma_2(u) & \text{if } u \in V_2 - V_1 \\
\sigma_1(u) \vee \sigma_2(u) & \text{if } u \in V_1 \cap V_2
\end{cases}
\]

And

\[
(\mu_1 \cup \mu_2)uv =
\begin{cases} 
\mu_1(uv) & \text{if } uv \in E_1 - E_2 \\
\mu_2(uv) & \text{if } uv \in E_2 - E_1 \\
\mu_1(uv) \vee \mu_2(uv) & \text{if } uv \in E_1 \cap E_2 \\
0 & \text{otherwise}
\end{cases}
\]

Proposition 2.2 [14] Let \( G \) be the union of the graphs \( G_1 \) and \( G_2 \) then \( G = G_1 \cup G_2: (\sigma_2 \cup \sigma_2, \mu_1 \cup \mu_2) \) is a product fuzzy graph.
Proposition 2.3 [14] Let G be the join of two graphs $G_1$ and $G_2$ and let $G_2: (\sigma_2, \mu_2)$ be the product fuzzy graph of $G_1 \& G_2$ resp.then $G = G_1 + G_2: (\sigma_2 + \sigma_1, \mu_2 + \mu_1)$ is a product fuzzy graph of G.

Theorem 2.1 [14] Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be the product fuzzy graphs then

i) $$(G_1 \cup G_2) \supseteq G_1 \cup G_2$$

ii) $$(G_1 + G_2) \supseteq G_1 \cup G_2$$

3. MAIN RESULTS

Definition 3.1 Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be the product fuzzy graphs with $G_1: (V_1, E_1)$ and $G_2: (V_2, E_2)$ res. Then the ring sum of two product fuzzy graphs $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ is denoted by $G = G_1 \oplus G_2: (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)$ where

$$(\sigma_1 \oplus \sigma_2)u = (\sigma_1 \cup \sigma_2)u \quad \forall u \in V_1 \cup V_2$$

And

$$(\mu_1 \oplus \mu_2)_{uv} = \begin{cases} 
\mu_1(\mu_1) & \text{if } uv \in E_1 - E_2 \\
\mu_2(\mu_1) & \text{if } uv \in E_2 - E_1 \\
0 & \text{otherwise}
\end{cases}$$

Proposition 3.1 Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be the product fuzzy graphs with $G_1: (V_1, E_1)$ and $G_2: (V_2, E_2)$ respectively. Then the ring sum of two product fuzzy graphs $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ is denoted by $G = G_1 \oplus G_2: (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)$ is a product fuzzy graph.

Proof: Since $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ are the product fuzzy graphs with $G_1: (V_1, E_1)$ and $G_2: (V_2, E_2)$ res. Then we have to show that the ring sum of two product fuzzy $G = G_1 \oplus G_2: (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)$ is a product fuzzy graph

For this we show that
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$$(\mu_1 \oplus \mu_2)_{uv} \leq \{(\sigma_1 \oplus \sigma_2)_{u} \times (\sigma_2 \oplus \sigma_3)_{v}\}$$ in all cases.

Case-1 \textbf{If} $uv \in E_1 - E_2$ and $u, v \in V_1 - V_2$ then

$$(\mu_1 \oplus \mu_2)_{uv} = \mu_1(uv)$$

$$\leq [\sigma_2(u) \times \sigma_3(v)]$$

$$= [(\sigma_2 \cup \sigma_3)_{u} \times (\sigma_2 \cup \sigma_3)_{v}] \quad \forall u, v \in V_1 - V_2$$

Case-2 \textbf{If} $uv \in E_1 - E_2$ and $u \in V_1 - V_2, v \in V_1 \cap V_2$ then

$$(\mu_1 \oplus \mu_2)_{uv} = \mu_1(uv)$$

$$\leq [(\sigma_2 \cup \sigma_3)_{u} \times \max(\sigma_2(v), \sigma_3(v))_{v}]$$

$$= [(\sigma_2 \cup \sigma_3)_{u} \times (\sigma_2 \cup \sigma_3)_{v}]$$

$$= [(\sigma_2 \oplus \sigma_3)_{u} \times (\sigma_2 \oplus \sigma_3)_{v}]$$

i.e. $$(\mu_1 \oplus \mu_2)_{uv} \leq [\max(\sigma_2(u), \sigma_3(u)) \times \max(\sigma_2(v), \sigma_3(v))_{v}]$$

Case-3 \textbf{If} $uv \in E_1 - E_2$ and $u, v \in V_1 \cap V_2$ then

$$(\mu_1 \oplus \mu_2)_{uv} = \mu_1(uv)$$

$$\leq [\max(\sigma_1(u), \sigma_3(u)) \times \max(\sigma_1(v), \sigma_3(v))_{v}]$$

$$= [(\sigma_2 \cup \sigma_3)_{u} \times (\sigma_2 \cup \sigma_3)_{v}]$$

$$= [(\sigma_2 \oplus \sigma_3)_{u} \times (\sigma_2 \oplus \sigma_3)_{v}]$$
i.e. \((\mu_1 \oplus \mu_2)_{uv} \leq [(\sigma_2 \oplus \sigma_2)u \times (\sigma_2 \oplus \sigma_2)v]\)

Similarly we can show that if \(uv \in E_2 - E_1\) then

\((\mu_1 \oplus \mu_2)_{uv} \leq [(\sigma_2 \oplus \sigma_2)u \times (\sigma_2 \oplus \sigma_2)v]\) for all \(u, v\).

This completes the proof of the proposition.

Example 3.1

![Diagram 1](image1.png)

\(G_1 : (\sigma_2, \mu_2)\)

![Diagram 2](image2.png)

\(G_2 : (\sigma_2, \mu_2)\)

![Diagram 3](image3.png)

\(G_1 \square G_2 : (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)\)

Remark 3.1 If \(G = G_1 \square G_2 : (\sigma_1 \oplus \sigma_2, \mu_1 \oplus \mu_2)\) is a product fuzzy graph. Then

\[\overline{(\sigma_2 \oplus \sigma_2)u} = (\overline{(\sigma_2 \oplus \sigma_2)u}) = (\sigma_2 \oplus \sigma_2)u\]

And

\[\overline{(\mu_1 \oplus \mu_2)_{uv}} = (\overline{(\sigma_2 \oplus \sigma_2)u}) \times (\overline{(\sigma_2 \oplus \sigma_2)v}) - (\overline{(\mu_1 \oplus \mu_2)_{uv}})\]
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\[ (\sigma_1 \oplus \sigma_2)u \times (\sigma_2 \oplus \sigma_3) = [((\sigma_1 \oplus \sigma_2)u \times (\sigma_2 \oplus \sigma_3)v) - (\mu_1 \oplus \mu_2)uv] \]

\[ = (\mu_1 \oplus \mu_2)uv \]

i.e. \( (\mu_1 \oplus \mu_2)uv = (\mu_1 \oplus \mu_2)uv \)

**Theorem 3.1** Let \( G_1 (\sigma_1, \mu_1) \) and \( G_2 (\sigma_2, \mu_2) \) be two product fuzzy graphs with \( E_1 \cap E_2 \neq \emptyset \) then

\[ \overline{E_1 \oplus E_2} \cong \overline{E_1} + \overline{E_2} \]

Proof:- Let \( G_1 : (\sigma_1, \mu_1) \) and \( G_2 : (\sigma_2, \mu_2) \) be two product fuzzy graphs and

\[ \Pi : V_1 + V_2 \rightarrow V_1 + V_2 \]

be the identity mapping, then it is sufficient to show that

\[ (\sigma_2 \oplus \sigma_3)u = (\sigma_1 + \sigma_2)u \]

And \( (\mu_1 \oplus \mu_2)uv = (\mu_1 + \mu_2)uv \)

Now, \( \sigma_2 \oplus \sigma_3 \) is a product fuzzy graph.

\[ \overline{(\sigma_2 \oplus \sigma_3)}u = (\sigma_1 \oplus \sigma_3)u \]

\[ = (\sigma_1 \cup \sigma_3)u \]

\[ = (\sigma_1 + \sigma_2)u \]

i.e. \( \overline{(\sigma_2 \oplus \sigma_3)}u = (\sigma_1 + \sigma_2)u \) \hspace{1cm} ............................................(i)

And \( (\mu_1 \oplus \mu_2)uv = ((\sigma_1 \oplus \sigma_2)u) \times ((\sigma_1 \oplus \sigma_2)v) - (\mu_1 \oplus \mu_2)uv \)

\[ = (\sigma_1 + \sigma_2)u \times (\sigma_1 + \sigma_2)v - [(\mu_1 \oplus \mu_2)uv] \]
Next we illustrate different cases

Case-1 When \(uv \in E_1\) i.e. \(u \in V_1\) and \(v \in V_2\)

\[
\overline{\mu_1 \oplus \mu_2}(uv) = \left( (\overline{\sigma_1 \oplus \sigma_2})u \times (\overline{\sigma_1 \oplus \sigma_2})v \right) - (\overline{\mu_1 \oplus \mu_2})uv
\]

\[
= (\overline{\sigma_1 \oplus \sigma_2})u \times (\overline{\sigma_1 \oplus \sigma_2})v - 0
\]

\[
= \overline{\sigma_1}(u) \times \overline{\sigma_2}(v)
\]

i.e. \(\overline{\mu_1 \oplus \mu_2}(uv) = \overline{\sigma_1}(u) \times \overline{\sigma_2}(v)\)

Case-2 When \(uv \in E_1 - E_2\) and \(u, v \in V_1 - V_2\)

\[
\overline{\mu_1 \oplus \mu_2}(uv) = [\sigma_1(u) \times \sigma_2(v)] - \mu_1(uv)
\]

\[
= \overline{\mu_1}(uv)
\]

i.e. \(\overline{\mu_1 \oplus \mu_2}(uv) = \overline{\mu_1}(uv)\) \(\forall uv \in E_1 - E_2\)

Case-3 When \(uv \in E_1 - E_2\) and \(u, v \in V_1 \cap V_2\)

\[
\overline{\mu_1 \oplus \mu_2}(uv) = [\sigma_1(u) \cup \sigma_2(v)] \times [\sigma_1(v) \times \sigma_2(v)] - \mu_1(uv)
\]

\[
= [(\sigma_1 \cup \sigma_2)u \times (\sigma_1 \cup \sigma_2)v] - \mu_1(uv)
\]

\[
= \overline{\mu_1}(uv)
\]

\(\because (\sigma_1 \cup \sigma_2, \mu_2)\) is product fuzzy graph

i.e. \(\overline{\mu_1 \oplus \mu_2}(uv) = \overline{\mu_1}(uv)\)
Case-4 When \( uv \in E_2 - E_1 \) and \( u \in V_1 - V_2, v \in V_1 \cap V_2 \)

\[
(\mu_1 \oplus \mu_2)uv = \sigma_1(u) \times (\sigma_2(u) \cup \sigma_2(v)) - \mu_1(\mu uv)
\]

\[
= [\sigma_1(u) \cup \sigma_2(u)] \times [\sigma_1(v) \times \sigma_2(v)] - \mu_1(uv)
\]

\[
= [(\sigma_1 \cup \sigma_2)u \times (\sigma_2 \cup \sigma_2)v] - \mu_1(uv)
\]

\[
= \mu_2(\mu uv) \quad \forall (\sigma_1 \cup \sigma_2, \mu_1) \text{ is product fuzzy graph}
\]

i.e.

\[
(\mu_1 \oplus \mu_2)uv = \mu_2(\mu uv)
\]

Case-5 When \( uv \in E_2 - E_1 \) and \( u, v \in V_2 - V_1 \) then it is obvious that

\[
\mu_1(\mu_2)uv = \mu_2(\mu uv)
\]

Case-6 When \( uv \in E_2 - E_1 \) and \( u, v \in V_1 \cap V_2 \) then it is obvious that

\[
\mu_1(\mu_2)uv = \mu_2(\mu uv)
\]

Case-7 When \( uv \in E_2 - E_1 \) and \( u \in V_2 - V_1, v \in V_1 \cap V_2 \) then it is obvious that

\[
\mu_1(\mu_2)uv = \mu_2(\mu uv)
\]

Case-8 When \( uv \in E_1 \cap E_2 \) then

\[
\mu_1(\mu_2)uv = (1_{\sigma_1 \oplus \sigma_2})u \times (\sigma_2 \cup \sigma_2)v - (\mu_1 \oplus \mu_2)uv
\]

\[
= [(\sigma_1 \cup \sigma_2)u \times (\sigma_2 \cup \sigma_2)v] - (\mu_1 \cup \mu_2)uv
\]

\[
= [(\sigma_2 \cup \sigma_2)u \times (\sigma_2 \cup \sigma_2)v] - (\mu_1(uv) \cup \mu_2(uv))
\]
\[ = \mu_1^{\oplus}(uv) \cup \mu_2^{\oplus}(uv) \]

i.e. \( \mu_1^{\oplus}(uv) \cup \mu_2^{\oplus}(uv) \) for all \( uv \in E_1 \cap E_2 \)

Thus from Case-1 to Case-8 we have

\[
\mu_1^{\oplus}(uv) = \begin{cases} 
\mu_1(uv) & \text{if } uv \in E_1 - E_2 \\
\mu_2(uv) & \text{if } uv \in E_2 - E_1 \\
[\mu_1(uv) \cup \mu_2(uv)] & \text{if } uv \in E_1 \cap E_2 \\
[\sigma_2(u) \times \sigma_2(v)] & \text{if } uv \in E' 
\end{cases}
\]

\[ = \overline{\mu_1 + \mu_2}(uv) \]

i.e. \( \overline{\mu_1 + \mu_2}(uv) = \mu_1 + \mu_2 \) \( uv \) .................................(ii)

Hence by (i) and (ii) we have

\[ \overline{G_1 \oplus G_2} \cong \overline{G_1} + \overline{G_2} \]

This completes the proof of the theorem.

**Theorem 3.2** Let \( G_1; \sigma_1, \mu_1 \) and \( G_2; \sigma_2, \mu_2 \) be two product fuzzy graphs with \( E_1 \cap E_2 = \emptyset \) then

\[ \overline{G_1 + G_2} \cong \overline{G_1 \oplus G_2} \]

Proof:- Let \( G_1; \sigma_1, \mu_1 \) and \( G_2; \sigma_2, \mu_2 \) be two product fuzzy graphs with \( E_1 \cap E_2 = \emptyset \) and

\[ I: V_1 + V_2 \rightarrow V_1 + V_2 \]

be the identity mapping, then it is sufficient to show that

\[ (\sigma_2 + \sigma_2)u = (\sigma_1 \oplus \sigma_2)u \]

And \( (\mu_1 + \mu_2)uv = (\mu_1 \oplus \mu_2)uv \)
Now,
\[ \overline{\sigma_1 + \sigma_2}u = (\sigma_1 + \sigma_2)u \]
\[ = (\sigma_1 \cup \sigma_2)u \]
\[ = (\overline{\sigma_1 \cup \sigma_2})u \]

i.e. \[ \overline{\sigma_1 + \sigma_2}u = (\overline{\sigma_1 \cup \sigma_2})u \] \hspace{1cm} (i)

Next \( (\overline{\mu_1 + \mu_2})uv = \left( (\sigma_1 + \sigma_2)u \times (\sigma_1 + \sigma_2)v \right) - (\mu_1 + \mu_2)uv \)

\[ (\overline{\mu_1 + \mu_2})uv = \begin{cases} \left[ ((\sigma_1 \cup \sigma_2)u \times (\sigma_1 \cup \sigma_2)v) - (\mu_1 \cup \mu_2)uv \right] & \text{if } uv \in E_1 \cup E_2 \\ \left[ ((\sigma_1(u) \times \sigma_2(v)) - (\sigma_1(u) \times \sigma_2(v)) \right] & \text{if } uv \in V_1 \cup V_2 \end{cases} \]

Next we illustrate different cases

Case-1 When \( uv \in E_1 \) i.e. \( u \in V_1 \) and \( v \in V_2 \)

\[ (\overline{\mu_1 + \mu_2})uv = \left[ (\sigma_1(u) \times \sigma_2(v)) - (\mu_1(u) \times \mu_2(v)) \right] \]
\[ = 0 \]

i.e. \[ (\overline{\mu_1 + \mu_2})uv = 0 \]

Case-2 When \( uv \in E_1 - E_2 \) and \( u, v \in V_1 - V_2 \)

\[ (\overline{\mu_1 + \mu_2})uv = \left[ (\sigma_1(u) \times \sigma_2(v)) \right] - \mu_1(u\bar{v}) \]
\[ = \overline{\mu_1}(uv) \]

i.e. \[ (\overline{\mu_1 + \mu_2})uv = \overline{\mu_1}(uv) \] \hspace{1cm} \forall uv \in E_1 - E_2

Case-3 When \( uv \in E_1 - E_2 \) and \( u, v \in V_1 \cap V_2 \)

\[ (\overline{\mu_1 + \mu_2})uv = \left[ ((\sigma_1 \cup \sigma_2)u \times (\sigma_1 \cup \sigma_2)v) \right] - \mu_1(u\bar{v}) \]
\( \mu_{1}(uv) \quad \forall (\sigma_{1} \cup \sigma_{2}, \mu_{1}) \) is product fuzzy graph

i.e. \( (\mu_{1} + \mu_{2})uv = \mu_{2}(uv) \)

Case-4 When \( uv \in E_{1} - E_{2} \) and \( u \in V_{1} - V_{2} \), \( v \in V_{1} \cap V_{2} \)

\[ (\mu_{1} + \mu_{2})uv = \sigma_{1}(u) \times (\sigma_{1}(v) \cup \sigma_{2}(v)) - \mu_{1}(uv) \]

\[ = [\sigma_{1}(u) \cup \sigma_{2}(u)] \times [\sigma_{1}(v) \times \sigma_{2}(v)] - \mu_{2}(uv) \]

\[ = [(\sigma_{1} \cup \sigma_{2})u \times (\sigma_{1} \cup \sigma_{2})v] - \mu_{2}(uv) \]

\[ = \frac{\mu_{2}(uv)}{\nu} \quad \forall (\sigma_{1} \cup \sigma_{2}, \mu_{1}) \) is product fuzzy graph

i.e. \( (\mu_{1} + \mu_{2})uv = \mu_{2}(uv) \)

Case-5 When \( uv \in E_{2} - E_{1} \) and \( u, v \in V_{2} - V_{1} \) then it is obvious that

\[ \mu_{1} + \mu_{2})uv = \mu_{2}(uv) \]

Case-6 When \( uv \in E_{2} - E_{1} \) and \( u, v \in V_{1} \cap V_{2} \) then it is obvious that

\[ \mu_{1} + \mu_{2})uv = \mu_{2}(uv) \]

Case-7 When \( uv \in E_{2} - E_{1} \) and \( u \in V_{2} - V_{1} \), \( v \in V_{1} \cap V_{2} \) then it is obvious that

\[ (\mu_{1} + \mu_{2})uv = \mu_{2}(uv) \]

Thus from Case-1 and Case-7 we have

\[ (\mu_{1} + \mu_{2})uv = \begin{cases} \mu_{1}(uv) & \text{if } uv \in E_{1} - E_{2} \\ \mu_{2}(uv) & \text{if } uv \in E_{2} - E_{1} \\ 0 & \text{otherwise} \end{cases} \]
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\[ = (\mu_1 \oplus \mu_2)_{uv} \]

i.e \( \mu_1 + \mu_2 \)_{uv} = \( \mu_1 \oplus \mu_2 \)_{uv} ..........................................................(ii)

Hence from (i) and (ii)

\[ G_1 + G_2 \cong G_1 \oplus G_2 \]

This completes the proof of the theorem.

**Theorem 3.3** If \( G \) is the Ring sum of two subgraphs \( G_1 \) and \( G_2 \) with \( E_1 \cap E_2 = \emptyset \), then every complete product fuzzy subgraph \((\sigma, \mu)\) of \( G \) is a ring sum of complete product fuzzy subgraph of \( G_1 \) and complete product fuzzy subgraph of \( G_2 \).

Proof:- We define the fuzzy subset \( \sigma_1 \), \( \sigma_2 \), \( \mu_1 \) and \( \mu_2 \) of \( V_1 \), \( V_2 \), \( E_1 \) and \( E_2 \) as follows

\[ \sigma_2(u) = \sigma(u) \text{ if } u \in V_2 - V_1 \text{, } \sigma_2(u) = \sigma(u) \text{ if } u \in V_1 - V_2 \]

And \( \mu_1(\mu v) = \mu(\mu v) \) if \( \mu v \in E_1 - E_2 \), \( \mu_2(\mu v) = \mu(\mu v) \) if \( \mu v \in E_2 - E_1 \)

Then \((\sigma_1, \mu_1)\) is product fuzzy graph of \( G_1 \) and \((\sigma_2, \mu_2)\) is product fuzzy graph of \( G_2 \).

And \( \sigma = (\sigma_1 \oplus \sigma_2) \) by definition of ring sum of product fuzzy graph.

If \( \mu v \in E_1 \cup E_2 \) then \( \mu(\mu v) = (\mu_1 \oplus \mu_2)_{\mu v} \) by definition of ring sum of product fuzzy graph.

If \( \mu v \in E_1 - E_2 \), \( \mu(\mu v) = (\mu_1 \oplus \mu_2)_{\mu v} \) by definition of ring sum of product fuzzy graph.

If \( \mu v \in E_2 - E_1 \), \( \mu(\mu v) = (\mu_1 \oplus \mu_2)_{\mu v} \) by definition of ring sum of product fuzzy graph.
If \( uv \in E \), i.e, \( u \in V_1 \) and \( v \in V_2 \), then \( (\mu_1 \oplus \mu_2)_{uv} = 0 = \mu(uv) \). And other equalities are also holds because \((\phi, \mu)\) is a complete product fuzzy graph.

**CONCLUSIONS**

We have discussed product fuzzy graph and its complement. The notion of a Ring sum and join of product fuzzy graphs are discussed. In the notion of ring sum and join of product fuzzy graphs common vertices and edges are considered. We have also discussed isomorphism property of product fuzzy graph which are fuzzy version of classical graphs.

**REFERENCES**


Some Isomorphism Results on Product Fuzzy Graphs


