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H-FACTOR AND *F*-FACTOR OF BUTTERFLY NETWORK

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ABSTRACT

For an arbitrary graph *G* and a fixed graph *H*, an *H*-packing of *G* is a set of vertex-disjoint subgraphs of *G*, each isomorphic to *H*. An *F*-packing of *G* is a set of pairwise vertex-disjoint subgraphs of *G*, each isomorphic to a graph in the family *F*. In this paper, we obtain an *H*-packing of odd dimensional Butterfly network and its packing number. Also *F*-packing of even dimensional Butterfly network is obtained.

KEYWORDS: *H*-packing, binding diamonds, partition, *F*-packing.

INTRODUCTION

Interconnection networks are becoming increasingly pervasive in many different applications, with the operational costs and characteristics of these networks depending considerably on the application. An interconnection network consists of hardware and software entities that are interconnected to facilitate efficient computation and communication. These entities can be in the form of processors, memory modules or computer systems. Interconnection networks play a key role in the design and implementation of communication networks and the recent advent of optic technology adds more design problems. Communication speed, high robustness, rich structure, fault tolerance, fixed degree are crucial criteria in the design of interconnection networks. Numerous parallel computer architectures meeting the several conflicting demands have been designed [1,2].

The matching problem of undirected graphs is a field of central importance in combinatorial optimization as it led to the development of fundamental techniques. It has a vast theory comprising of well known results with applications in all parts of combinatorics. The fundamentals of this field were established to a large extent by pioneering works of Tutte, Gallai, Edmonds and Lovasz [3,4]. The notion of matching in a graph also has also numerous applications in diverse areas such as traversal theory, network flow and multiprocessor design [3,5].

Given G and H, an *H-packing* of G is a collection of vertex-disjoint copies of H in G. In other words, the *H-packing* of G is a set $\{H_1, H_2, H_3, H_4, \ldots\}$ of vertex-disjoint subgraphs of G where each subgraph is isomorphic to H. An *H-packing* covers a vertex v of G if one of the subgraphs of the packing contains v. An H-packing of G is maximum, if it covers the greatest possible number of vertices of G and is called a *perfect H-packing* or an *H-factor* of G, if it covers all the vertices of G. The *H-packing number* of G is the maximum cardinality of the H-packing of G.

Let $F = \{H_1, H_2, H_3, H_4, \ldots\}$ be a family of graphs. An F-packing of G is a set of pairwise vertex-disjoint subgraphs of G, each isomorphic to a graph in the family F. We say that an F-packing covers a vertex of G if one of the subgraphs in the family of the packing contains that vertex. If an F-packing covers all the vertices of G, then it is a *perfect F*-packing, also called an *F-factor* of G[4,5].

A matching in G may be viewed as a collection of disjoint subgraphs of G, each isomorphic to K_2. In a *perfect matching*, the vertex set is completely partitioned by the vertex sets of the subgraphs.

When H has only components isomorphic to K_1 and K_2, then the maximum H-packing problem becomes the familiar maximum matching problem in bipartite graphs which can be solved in polynomial time using the matching algorithm [6]. Over the past four decades, many research works have been pursued in packing of graphs [4]. When the graph H is a connected graph with at least three vertices, D.G. Kirkpatrick and P. Hell proved that the *H-packing problem*(*H-factor problem)* and *F-packing problem*(*F-factor problem)*is *NP-complete* [6,7].

Apart from theoretical interest, the graph packing problem is of practical interest in the areas of scheduling [6], wiring-board design, code optimization, exam scheduling and in the study of degree constraint subgraphs [9] and wireless sensor tracking [10].

AN OVERVIEW OF THE PAPER

Let $BF(r)$ be the r-dimensional Butterfly network. Let C_4 be the cycle of length four. Let $M_r(G, H)$ be the maximum number of vertex-disjoint subgraphs H of G called the H-packing number of G. By $C_4 \circ P_3$ we denote a cycle of length four and a path of length two identifying one vertex from each of degree two. From the optimization point of view, this paper focuses on H-factor of $BF(r)$, when r is odd and F-factor of $BF(r)$, when r is even.

TOPOLOGICAL PROPERTIES OF BUTTERFLY NETWORK

Butterfly network is an important and well known topological structure of interconnection networks. It is a bounded-degree derivative of the hypercube which aims at overcoming some drawbacks of hypercube. It is used to perform a method to illustrate FFT(Fast Fourier Transform), which is intensively used in the field of signal processing[2]. It is used in performing arbitrary permutation of a sequence in a processor and in the field of channel coding permutation and depermutation [1,2].

Networks are represented as undirected graphs whose nodes represent processors and whose edges represent interprocessors communication links. The set V of nodes of an r-dimensional butterfly correspond to pairs $[w, i]$, where i is the dimension or level of a node ($0 \leq i \leq r$) and w is an r-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either (1) w and w' are identical, or (2) w and w' differ in precisely the i^{th} bit. The edges in the network are undirected. The r-dimensional butterfly is denoted by $BF(r)$. It has $2^r(r + 1)$ vertices and $r2^{r+1}$ edges [1,2].

Manuel et al. [11] have identified a new topological representation of butterfly networks and proposed the new representation as the *diamond representation* and have proved that the normal and diamond representations of butterfly network are isomorphic. Figure 1 is the diamond representation of a 3-dimensional butterfly network.

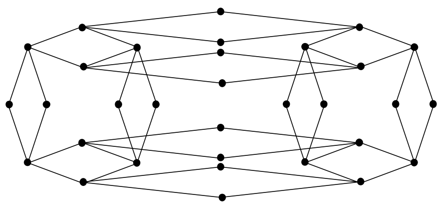

Figure 1 : Diamond Representation of BF(3)

The diamond representation of butterfly network is defined as follows: Two $(r-1)$-dimensional butterfly networks $BF(r-1)$ form mirror images with respect to an array of level 0 nodes. The level 0 nodes are the vertices belonging to chordless 4-cycles in the diamond formation bridging the two $(r-1)$-dimensional butterfly networks $BF(r-1)$. Each 4-cycle is drawn as a diamond. This representation provides a structural visualization and an in-depth understanding about the cyclic properties and the organization of spanning trees of butterfly.

Two nodes $[w, i]$ and $[w', i']$ are said to be mirror images of each other if w and w′ differ precisely in the first bit. The removal of level 0 vertices $\{v_1, v_2, \ldots, v_{2^r}\}$ of $BF(r)$ gives two subgraphs H_1 and H_2 of $BF(r)$, each isomorphic to

BF(*r*-1). Since $\{v_1, v_2, \ldots, v_{2^r}\}$ is a vertex-cut of *BF*(*r*), the vertices are called binding vertices of *BF*(*r*). If a 4-cycle in *BF*(*r*) has binding vertices then it is called a binding diamond. The edges of binding diamonds are called binding edges[8]. We face a similar situation when vertices of *BF*(*r*) at level (n + 1) are removed. To distinguish between the two, we call the binding diamonds defined by removing vertices at level 0 as vertical binding diamonds and those defined by removing vertices at level (n + 1) as horizontal binding diamonds. See Figure 2.

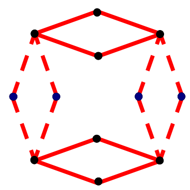

Figure 2 : Horizontal and Vertical Binding Diamonds of *BF*(2)

Butterfly networks are among the most important topologies for building commercial and experimental parallel computers, special purpose processors, network switches. This network has been studied extensively and has served as the routing network in several parallel computers and ATM switches. The butterfly network (*k*-ary *n*-fly) can take advantage of high-radix routers to reduce latency and network cost[2]. The number of vertical binding diamonds in an *r*-dimensional butterfly network *BF*(*r*) is 2^{r-1} and the number of horizontal binding diamonds is also 2^{r-1}.

Theorem 1: Let *G* be an odd dimensional butterfly network *BF*(*r*) and *H* be a subgraph of *G*. If $H \cong C_4$, then there exists an *H*-factor of *BF*(*r*) with $M_r(G, H) = 2^{r-2}(r+1)$.

Proof: We prove this theorem by induction on the dimension of the butterfly network *BF*(*r*), when *r* is odd.

Base Case $r = 3$:_The total number of vertical and horizontal binding diamonds is $2^3 = 2^{3-2}(3 + 1) = M_3(G, H)$. Since $|V(BF(3))| = 32$, the packing of $BF(3)$ is perfect. See Figure 3.

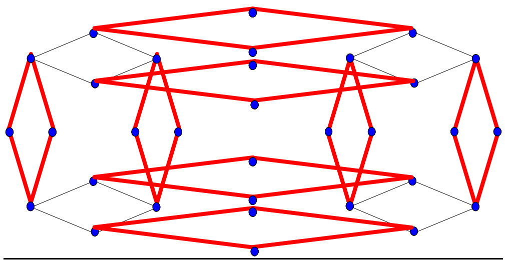

Figure 3 : H-factor of BF(3)

Suppose that the theorem is true for $BF(m)$ where $1 \leq m \leq r$ and m is odd.

Consider $BF(r + 2)$. The total number of vertical and horizontal binding diamonds is 2^{r+2}. These binding diamonds partition $V(BF(r + 2))$ into 16 sets, each inducing an $(r – 2)$-dimensional butterfly network $BF(r – 2)$. By induction hypothesis, $M_{r-2}(G, H) = 2^{r-2-2}(r – 2 + 1) = 2^{r-4}(r – 1)$. Thus, $M_{r+2}(G, H) = 2^{r+2}+16[2^{r-4}(r – 1)] = 2^r (r + 3)$. Since $|V(BF(r))| = 2^{r+2}(r + 3)$, the packing of $BF(r + 2)$ is perfect. Hence the theorem.

Corollary: Let G be an even dimensional butterfly network $BF(r)$ and $H \cong C_4$. In any H-packing of $BF(r)$, there are 2^r isolated vertices.

Proof: We prove this by induction on the dimension of butterfly network $BF(r)$, when r is even.

Base Case $r = 2$: In $BF(2)$, there are exactly two vertex disjoint 4-cycles. The remaining are 4 isolated vertices. See Figure 4.

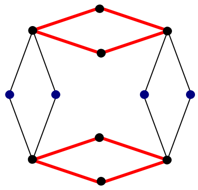

Figure 4 : Isolated vertices of *BF*(2)

Suppose that the theorem is true for $BF(m)$, $1 \leq m \leq r$ and m is even

Consider $BF(r + 2)$. The total number of vertical and horizontal binding diamonds is 2^{r+2}. These binding diamonds partition $V(BF(r + 2))$ into 16 sets, each inducing an $(r - 2)$-dimensional butterfly network $BF(r - 2)$. By induction hypothesis on r, $BF(r - 2)$ has 2^{r-2} isolated vertices. Hence, $BF(r + 2)$ has $16 \times 2^{r-2} = 2^{r+2}$ isolated vertices.

Open Problem: Is there a graph H such that $BF(r)$, r even, admits an H-factor?

Hence, we consider a family F of two subgraphs, $F = \{H_1, H_2\}$ where $H_1 \cong C_4$ and $H_2 \cong C_4 \circ P_3$. If $r = 2$, then two sets of $C_4 \circ P_3$ is a perfect packing of $BF(r)$. See Figure 5.

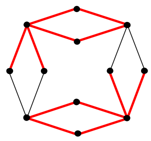

Figure 5 : Packing BF(2) with $C_4 \circ P_3$

We consider two cases on the dimension of butterfly network r (i) $r \equiv 2 (\text{mod } 4)$ (ii) $r \equiv 0 (\text{mod } 4)$

Theorem 2: Let G be an even dimensional butterfly network $BF(r)$ where $r \equiv 2(\text{mod } 4)$. If $H_1 \cong C_4$ and $H_2 \cong C_4 \circ P_3$, then there exists an F-factor of $BF(r)$.

Proof: We use induction on the dimension of butterfly network $BF(r)$, where $r \equiv 2 (\text{mod } 4)$.

Base Case $r = 6$: The total number of vertical and horizontal binding diamonds is 2^6. These binding diamonds partition $V(BF(6))$ into 16 sets, each inducing $BF(2)$ and $C_4 \circ P_3$ is a perfect packing of $BF(2)$. Since, $|V(BF(6))| = 4\left\{2^6 \left\lfloor\frac{6}{4}\right\rfloor\right\} + 6\{2^{6-1}\} = 448 = 2^6(6+1)$, the packing of $BF(6)$ is perfect.

Assume that the theorem is true for $BF(m)$, $1 \leq m \leq r$ and r even.

Consider $BF(r + 2)$. The total number of vertical and horizontal binding diamonds are 2^{r+2}. These binding diamonds partition $V(BF(r + 2))$ into 16 sets, each inducing an $(r - 2)$-dimensional butterfly network $BF(r - 2)$. By induction on r, $|V(BF(r-2)| = 4\left\{2^{r-2} \left\lfloor\frac{r-2}{4}\right\rfloor\right\} + 6\{2^{r-3}\}$. Since, $|V[BF(r + 2)]| = 4(2^{r+2}) + 16\{|V[BF(r-2]|\} = 4\left\{2^{r+2} \left\lfloor\frac{r+2}{4}\right\rfloor\right\} + 6\{2^{r+1}\} = 2^{r+2}(r+3)$, .the packing of $BF(r + 2)$ is perfect. Hence the theorem.

Theorem 3: Let G be an even dimensional butterfly network $BF(r)$ where $r \equiv 0 (\text{mod } 4)$. If $H_1 \cong C_4$ and $H_2 \cong C_4 \circ P_3$, then there exists an F-factor of $BF(r)$.

Proof: We use induction on the dimension of Butterfly network $BF(r)$, where $r \equiv 0 (\text{mod } 4)$.

Base Case $r = 4$: The total number of horizontal binding diamonds is 2^3. These binding diamonds partition $V[BF(6)]$ into 4 sets, each inducing $BF(2)$ and $C_4 \circ P_3$ is a perfect packing of $BF(2)$. Since, $|V[BF(4)]| = 4\{2^2(2)\} + 6\{2^3\}. = 80 = 2^4(4+1)$, the packing of $BF(4)$ is perfect. See Figure 6.

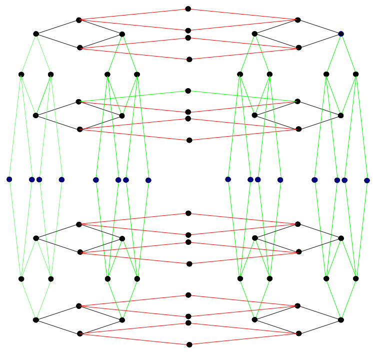

Figure 7 : Packing $BF(4)$ with C_4 and $C_4 \circ P_3$

Assume that the theorem is true for $BF(m)$, $1 \leq m \leq r$ and r even.

Consider $BF(r + 2)$. The total number of vertical and horizontal binding diamonds are 2^{r+2}. These binding diamonds partition $V(BF((r + 2))$ into 16 sets, each inducing an $(r – 2)$-dimensional butterfly network $BF(r - 2)$. By induction on r, $|V(BF(r -2)| = 4\{2^{r-4}(r-4)\} + 6\{2^{r-3}\} = 2^{r-2}(r-1)$. Since, $|V(BF(r + 2))| = 4(2^{r+2}) + 16\{|V[BF(r - 2]|\} = 4\{r2^{r}\} + 6\{2^{r+1}\} = 2^{r+2}(r+3)$,.the packing of $BF(r + 2)$ is perfect. Hence the theorem.

CONCLUSIONS

In this paper, we have obtained the C_4-factor and the C_4-packing number of $BF(r)$, when r is odd. The $F = \{H_1, H_2\}$ -factor of even dimensional butterfly network where $H_1 \cong C_4$ and $H_2 \cong C_4 \circ P_3$ is also obtained. Packing of Benes network is under consideration.

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