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GENERALIZED GAUSSIAN QUADRATURE RULES OVER TWO-DIMENSIONAL REGIONS WITH A CURVED EXPONENTIAL EDGE

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ABSTRACT

This paper presents a generalized Gaussian quadrature method for numerical integration over two-dimensional bounded regions with a curved exponential edge. A general formulae for numerical integration over the regions $R_1 = \{(x,y)|a \leq x \leq b,\ c \leq y \leq e^{kx}\}$ and $R_2 = \{(x,y)|a \leq y \leq b,\ c \leq x \leq e^{ky}\}$ are derived, which can be directly used for integrating any type of functions over such regions. In order to derive this formulae, a general transformation of these regions is given from (x,y) space to a square in (ξ,η) space, $S:\{ (\xi,\eta) \mid 0 \leq \xi \leq 1,\ 0 \leq \eta \leq 1\}$. Generlized Gaussian quadrature nodes and weights introduced by Ma *et.al.* in 1997 are used in the product formula presented in this paper to evaluate the integral over S, as it is proved to give more accurate results than the classical Gauss Legendre nodes and weights. The method can be used to integrate a wide class of functions including smooth functions and functions with end-point singularities. The performance of the method is illustrated for different functions over different regions with numerical examples.

Keywords: Numerical integration, Quadrature rules, Gaussian quadrature, Finite-element method

1. INTRODUCTION

There are many functions appearing in real life which cannot be integrated analytically, especially if integration is over regions with curved boundaries. Numerical integration methods are the only solutions to this. From the literature review we may realize that lot of works in numerical integration using Gauss quadrature over triangular regions has been done [1-9]. However not many quadrature rules were derived over elements with curved edges. The present work aims to derive new quadrature rules over triangular and rectangular elements with elliptic edges. In [9] Rathod *et.al.* have derived new quadrature rules over a triangle by transforming the triangle to a standard 2-square. Gauss Legendre quadrature nodes and weights were used by them in [9] to derive the quadrature rules over the triangle. In our paper we transform the triangular or the rectangular region with exponential sides in (x,y) space to a square in (ξ,η) space, $\mathrm{S}:\{(\xi,\eta)\,|\,0\le\xi\le 1,\ 0\le\eta\le 1\}$.

The method proposed here is termed as generalized Gaussian quadrature rules, since the generalized Gaussian quadrature nodes and weights for products of polynomials and logarithmic function given in [10], by Ma. e*t al* are used in the product formula presented in this paper. These nodes and weights give better integral values than the Gauss Legendre nodes and weights, even for functions with singularities.

The paper is organized as follows. In section 2, we give basic information about the generalized Gaussian quadrature method. In section 3, we derive the generalized Gaussian quadrature rules over triangular / rectangular regions with an exponential edge and in section 4 numerical results are presented.

2. GAUSSIAN QUADRATURE

The Gaussian quadrature is a numerical integration formula given by

$$\int_a^b q(x)\phi(x)dx = \sum_{i=1}^{N} w_i \phi(x_i) \quad (1)$$

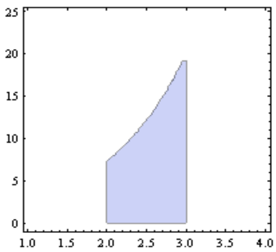
where $x_i \epsilon [a,b]$ and $w_i \epsilon \boldsymbol{R}$, for all $i = 1,2, \ldots, N.$ The points x_i and the coefficients w_i are referred to as the nodes and weights of the quadrature formula. The quadrature formula given in Eq.(1) is called a classical Gaussian quadrature rule if it integrates exactly all polynomials of order upto 2n-1, whereas Eq. (1) is said to be a generalized Gaussian quadrature rule with respect to a set of functions $\{\phi_1, \phi_2, \ldots, \phi_{2n}\}$ if it integrates exactly all the 2n functions in the set $\{\phi_1, \phi_2, \ldots, \phi_{2n}\}$.

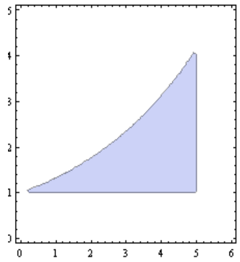
The weights and nodes (w_i, x_i) for the generalized Gaussian quadrature with respect to the set of functions $\{1, lnx, x, xlnx, \ldots, x^{N-1}, x^{N-1}lnx\}$ for N=5,10,15,20 are given in the table 1 of [10] along with the quadrature rule :

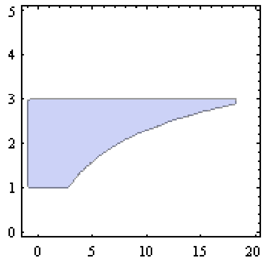
$$\int_0^1 \phi(x)dx = \sum_{i=1}^{N} w_i \phi(x_i) \quad (2)$$

3. GENERALIZED GAUSSIAN QUADRATURE OVER REGIONS WITH AN EXPONENTIAL EDGE

In this section we shall derive numerical integration formulae, which shall be called as generalised Gaussian quadrature rules for integrating functions over bounded triangular/rectangular shaped regions with one edge curved(exponential edge $y = e^{kx}$ or $x = e^{ky}$). Few such regions are plotted below







$2 < x < 3, 0 < y < \mathrm{Exp}[x]$ $0 < x < 5, 1 < y < \mathrm{Exp}[2x/7]$ $1 < y < 3, -1 < x < \mathrm{Exp}[y]$

Any such regions can be represented as R_1 or R_2, where,

$\mathbf{R_1} = \{(x,y)|a \leq x \leq b, c \leq y \leq e^{kx}\}$ and

$R_2 = \{(x,y)|a \leq y \leq b, c \leq x \leq e^{ky}\}$.

3.1 Formulation of integrals over $\mathbf{R_1 = \{(x,y)|a \leq x \leq b, c \leq y \leq e^{kx}\}}$

The integral of an arbitrary function, f, over the surface, $R_1 = \{(x,y)|a \leq x \leq b, c \leq y \leq e^{kx}\}$ is given by,

$$I = \int_a^b \int_c^{e^{kx}} f(x,y)\, dy\, dx \tag{3}$$

The integral I in Eq.(3) can be transformed into an integral over the surface of the square by the substitution:

$$x = (b-a)\xi + a\,; \tag{4}$$

Then the determinant of the Jacobian is

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = (b-a)\left[e^{k\{(b-a)\xi+a\}} - c\right] \tag{5}$$

We know that, in the given region,

(i) $b > a$ and hence $b - a > 0$

(ii) $e^{kx} > c$ and hence, $e^{k\{(b-a)\xi+a\}} > c$ (since $x = (b-a)\xi + a$)

Hence | J | in (5) is always positive and we have

$$dydx = |J|\, d\xi\, d\eta = (b-a)\left[e^{k\{(b-a)\xi+a\}} - c\right] d\xi\, d\eta \qquad (6)$$

Then, on applying Eq. (4) and (6) in Eq.(3), we get

$$I = \int_a^b \int_c^{e^{kx}} f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 f(x(\xi,\eta), y(\xi,\eta))\, |J|\, d\xi\, d\eta \qquad (7)$$

where x, y and $|J|$ are given in (4) and (5)

On applying the Gaussian quadrature rule given in Eq.(2), twice in the integral I in Eq.(7), we get

$$I = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j (b-a) \left[e^{k\{(b-a)\xi_i+a\}} - c\right] f\left(x(\xi_i,\eta_j), y(\xi_i,\eta_j)\right) \qquad (8)$$

where ξ_i, η_j are the generalized Gaussian quadrature points (taken from table 1 in [10]) in the ξ, η directions and w_i, w_j are the corresponding weights.

We re-write (8) as:

$$I = \sum_{k=1}^{N^2} c_k\, f(x_k, y_k) \qquad (9)$$

where

$$c_k = w_i w_j (b-a) \left[e^{k\{(b-a)\xi_i+a\}} - c\right];$$

$$x_k = (b-a)\xi_i + a;$$

$$y_k = \left[e^{k\{(b-a)\xi_i+a\}} - c\right]\eta_j + c \qquad (10)$$

The new weight coefficients c_k and the sampling points (x_k, y_k) of various orders can be computed easily using the formula in Eq. (10). Once the weight coefficients c_k and the sampling points (x_k, y_k) are obtained, the integral value can be calculated using the quadrature rule (9). Eq.(9) and (10) together gives the generalised Gaussian quadrature formulae over R_1.

3.2 Generalized Gaussian quadrature over $\mathbf{R_2 = \{(x, y) | a \leq y \leq b, c \leq x \leq e^{ky}\}}$

For integrating a function over the region $R_2 = \{(x,y) | a \leq y \leq b, c \leq x \leq e^{ky}\}$, we can derive the generalized Gaussian quadrature rules in a similar manner as in section 3.1 for R_1 .

The generalized Gaussian quadrature rule for an arbitrary function, f, over the region R_2 is given by:

$$I = \int_a^b \int_c^{e^{ky}} f(x, y)\, dx\, dy$$

$$= \sum_{k=1}^{N^2} c_k\, f(x_k, y_k) \qquad (11)$$

where

$$c_k = w_i w_j (b - a) \left[e^{k\{(b-a)\eta_j + a\}} - c\right];$$

$$x_k = \left[e^{k\{(b-a)\eta_j + a\}} - c\right]\xi_i + c;$$

$$y_k = (b - a)\eta_j + a; \qquad (12)$$

4. Numerical results

	Computed value
$\int_0^1 \int_0^{e^x} (1-y)\sin(10x)\, dy\,dx$ = 0.002693997109651	
N=5	0.042953152293338
N=10	0.002692243501226
N=15	0.002693997102977
N=20	0.002693997109651
$\int_1^3 \int_{-1}^{e^y} \sqrt{x+y}\, dx\,dy$= 49.4484656488193	
N=5	49.4466188093258
N=10	49.4484656464159
N=15	49.4484656488606
N=20	49.4484656488267
$\int_0^1 \int_0^{e^y} \sqrt{x^2+y^2}\, dx\,dy$=1.97907329225441	
N=5	1.97904996171999
N=10	1.97907327750052
N=15	1.97907329193809
N=20	1.97907329223719
$\int_0^1 \int_0^{e^y} [(x+y)^{1/2}(1+x+y)^2]\, dx\,dy$= 16.2596792004835	
N=5	16.2590357486570
N=10	16.2596792002626
N=15	16.2596792004652
N=20	16.2596792004827
$\int_1^2 \int_0^{e^x} (x^4+y^3)/(1+x^2y)\, dy\,dx$= 0.952005508874281	
N=5	0.952005503801037
N=10	0.952005508874288
N=15	0.952005508871023
N=20	0.952005508874288

$\int_1^3 \int_1^{e^{-y}} \sqrt{x^2+y^2}\, dxdy$ = -3.63492004187039	
N=5	-3.63491845047280
N=10	-3.63492004186677
N=15	-3.63492004185857
N=20	-3.63492004187040
$\int_2^3 \int_0^{e^{x}} (x^4+y^3)/(1+x^2y)\, dxdy$ = 145.062642914305	
N=5	145.389831960991
N=10	145.060063832437
N=15	145.062639597588
N=20	145.062643584708

5. CONCLUSIONS

In this paper we derive generalized Gaussian quadrature rules for calculating integrals over triangular and rectangular regions, with an exponential edge. The quadrature rules are given in general for regions R_1 and R_2. The results are obtained for different regions having different values of *a*, *b* and *k* in R_1 and R_2. Almost all the numerical results we have obtained using these new quadrature rules is exact for at least 10 decimal places. Also the quadrature formula derived here is not very complicated and hence doesn’t take much time for evaluation of the integrals. Any programming language or any mathematical software can be used for evaluating the results. The proposed method can be used to integrate a wide class of functions including functions with end-point singularities. It may be noted that integration over triangular and rectangular elements with straight and curved edges are common in finite-element method(FEM) and so the proposed quadrature rules due to its high precision can be applied to many problems in FEM.

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