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CHAOTIC MAPS ON MEASURE SPACES AND THE BEHAVIOR OF ORBITS OF STATES

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ABSTRACT:

We introduce maps with n laps on measure spaces and consider chaotic properties in those maps. Our purpose is to study behavior of orbit of probability density function instead of orbit of point. We show that a typical property of chaotic map converges to a unique function determined by the map in the case of orbit of probability density function though it is sensitive dependence on initial conditions in the case of orbit of point.

Keywords: measure, states, maps with n laps,Ferron-Frobenius, chaotic, endomorphism

INTRODUCTION

Chaotic maps are considered as those φ 's which have the following property (cf. [1]).

1. The set of all periodic points for φ are dense.
2. φ is transitive.
3. φ has sensitive dependence on initial conditions.

Those properties are concerned with the orbit of given initial point (cf. [9]). In this note, we consider how probability density functions are changed by iteration of chaotic Maps. More generally, we study behavior of states ω is changed by iterated *-endomorphisms $\{\alpha^k_{\nu(\varphi)}\}_{k=1}^{\infty}$ corresponding to iterated chaotic map φ^k. In particular, we show some theorems concerning the convergence in orbits of states. In those theorems, we can find that the following is an important property for chaotic map.

(1) The sequence of states $\{\alpha^k_{\nu(\varphi)}(\omega)\}_{k=1}^{\infty}$ converge to a unique state in the norm topology.

In section 1, we note our results concerning the relationship between *-endomorphisms of Von Neumann algebra and behavior of states transposed by iterated *-endomorphisms. In section 2, we note our main results concerning the relationship among chaotic maps, *-endomorphisms associated with those maps and of only examples which give us the meaning of theorems in Section 2 and provide fruitful discussion on or theory. Moreover we can find deep relation ship between our study and wavelet theory (cf. [3]). This note is a continuation of [5] and an expression of [4] and [5] by examples.

1. A *-ENDOMORPHISMS OS VON NEUMANN ALGEBRA ASSOCIATED WITH A FAMILY ISOMETRICS

Let H be a Hilbert space with inner product <.,.>. In this note $\{V_i\}_{i=1}^{n}$. means a family of isometrics on H satisfying the following property and is said to be a *f.i.c.* on H for short.

$\{V_i V_i^*\}_{i=1}^{n}$ is a set of mutually orthogonal projections and $\sum_{i=1}^{n} V_i V_i^* = 1$ (CP-1)

Of course, this family $\{V_i\}_{i=1}^{n}$ on H is the generators of the image of a representation of Cuntz-algebra O_n [2]. Moreover we can define a *-

endomorphisms *av* of the full operator algebra *B* (H) as follows: $\alpha v(T)=\sum_{i=1}^{n} V_i T V_i^*, (T\in B(H))$

If a von Neumann algebra M (cf.[8]) on is invariant for αv, then αv becomes a*- endomorphism of M. For n and a positive integer *k,* we denote by *I(n)* the set { 1,2,.....n} and $I(n)^k$ the set of k-tiuples, $\mu=(j_1,...,j_2)$ with j_1 in {1, 2 ...n}. For μ in $I(n)^k$ we denote by $V(\mu)$ the isometry $V_{j1},V_{j2},...,V_{jk}$ on (H). The is family of isometrics whose final projections are mutually orthogonal. When αv is a *- endomorphism of $M\alpha^k$ is of the form: $\alpha_v^k(T)=\sum_{\mu\in I(n)^k} V(\mu)TV(\mu)^*$

Proposition 1.1. Let $\{V_i\}_{i=1}^{n}$ be a f.i.c. on H and e a unit vector in H such that $V_i e = e$. We put

$$ONS(e,V)=\bigcup_{k=1}^{\infty}\{V(\mu)e \mid \mu\in I(n)^k\}$$

Then $ONS(e,V)$ is an orthogonal system.

Remark: An orthogonal system $ONS(e,V)$ in the above proposition is regarded as the sequence $\{e_k\}_{k=1}^{\infty}$ which is inductively defined as follows: $e_1=e$ and $e_{i+n(l-1)}=V_i e_l \qquad (i\in I(n), l\in N)$

For von Neumann algebra M on H, M* denotes the predual of M. we denote by α_v^* the transpose map of αv with respect to the duality of M and M*. the vector state in M* associated with unit vector ξ in H is denoted by $\omega\xi$, that is for T in

M. $\omega\xi(T) = < T\xi, \xi >$ and $\omega\xi(\alpha v(T)) = < \alpha v(T)\xi, \xi > = \alpha_v^*(\omega\xi)(T)$

Moreover, we have $\alpha_v^*(\omega\xi)\sum \omega V_i^*\xi$

When e is a unit vector such that $V_i e = e$ namely, it is an eigenvector for eigenvalue 1 of V_1 we denote by H_e the spanned by $ONS(e,V)$

Proposition 1. 2. Let $\{V_i\}_{i=1}^n$ be a f.i.c on H. If there exists a unit vector e such that $V_i e = e$, then for any init vector ξ in the subspace H_e it follows that $\lim_{n\to\infty}(\alpha_v^*)^n(\omega\xi) = \omega_e$

Proposition 1.3. Let $\{V_i\}_{i=1}^n$ be a f.i.c on H. If there exists a unit vector e such that $V_i e = e$, then for any state ω of the form $\omega = \sum \omega_{\xi_k}$ where ξ_k, s are in H_e, it follows that $\lim_{n\to\infty}(\alpha_v^*)^n(\omega) = \omega_e$.

Proposition 1.4. Let $\{V_i\}_{i=1}^n$ be a f.i.c on H. If there exists a unit vector e such that $V_i e = e$, if $ONS(e,V)$ is complete, then for any state ω in the predual of B(H) it follows that $\lim_{n\to\infty}(\alpha_v^*)^n(\omega) = \omega_e$.

Proposition 1.5. Let *M* Neumann algebra on H and $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ be a complete families of isometrics on H satisfying (1.1). Suppose that M is invariant for αv and αw, then the following conditions are equivalent.

(1) $\alpha v(T) = \alpha w(T)$ for all *T* in *M*.

(2) $$(W_1,..........,W_n)=(V_1,.......,V_2)\begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix},$$

That is , $W_i = \sum_{j=1}^{n} V_j h_{ji}, (1 \le i \le n)$,where h_{ji} is a unliterary element in the commutant M' of M on the Hilbert space *H*.

2. CHAOTIC MAPS AND BEHAVIOR OF STATES

Let X be measure space with measure ma and φ a measurable map on X, Here we note some notations concerning X and φ .

1. $mo\varphi$ denotes the measure on X defined by $mo\varphi(E) = m(\varphi(E))$ and if the map φ is absolutely continuous with respect to m, the Radon-Nikodym derivative for $mo\varphi$ and m is defined by $dmo\varphi / dm$
2. $\alpha\varphi$ denote the*-endomorphism of $L^{\infty}(X) = L^{\infty}(X,m)$ defined by $\alpha\varphi(f)(x) = f(\varphi(x))$ for f in $L^{\infty}(X)$.
3. T^{∞} denotes the linear operator on the Hilbert $H = L^2(X) = L^2(X, \text{m})$ space defined by $(T_{\varphi}\xi)(x) = \xi(\varphi(x)$ for ξ in H.
4. For a subset Y of *X*, x_x means the characteristic function of Y.
5. For a measurable function f on *X*, M_f *denotes* the multiplication operator on defined $L^2(x)$ by $M_f\xi = f\xi$ for ξ in $L^2(x)$.
6. For in $L^{\infty}(X), \pi(f)$ denotes the bounded multiplication operator on $L^2(x)$ defined by $\pi(f)\xi = f\xi$ for ξ in $L^2(x)$.

Definition 2.1.

Let X be measure space with measure m. A measurable map of X onto X is said to be a map with n-laps, MWnL for short, if there exists n measurable subsets $\{X_i\}_{i=1}^n$ of *X* such that

$$\bigcup_{i=1}^n X_i \text{ and } X_i \cap X_j = \varphi \text{ for } i \neq j$$

Each restriction φ_i of φ to X_i is a bimeasurable map of X_i on to X in the sense that φ_i is an subjective map of X_i on to φ_i X_i with $m(X \setminus \varphi_i(X_i)) = 0$ and φ_i^{-1} is measurable too.

For each i , $mo\varphi_i$ and $mo\varphi_i^{-1}$ are absolutely continuous with repect to m and non=singular in the sense that $\frac{dmo\varphi}{dm}(x) \neq 0$ a.e.x and $\frac{dmo\varphi^{-1}}{dm}(x) \neq 0$,a.e.x

For a measure space *(X, m)* and a measurable map φ of *X* into itself M_f , and T_φ is not necessarily defined on the full space *H*. Then each isometry V_i in the following definition if necessary is considered as a uniquely extended bounded linear operator on the full Hilbert space *H*.

$$\frac{dmo\varphi}{dm}(x) \neq 0 \text{ a.e.x and } \frac{dmo\varphi^{-1}}{dm}(x) \neq 0 \text{ ,a.e.x}$$

Definition 2.2.

Let φ be a MWnL on a measure space (X.m). We define a family isometrics $\{V_i(\varphi)\}_{i=1}^n$ associated with φ as follows : $V_i(\varphi) = M_{\sqrt{dmo\varphi/dm}} M_{\chi_{X_i}} T_\varphi (i = 1,\ldots n)$,

By the definition we can see that

(1) $V_i(\varphi)^* = M_{\sqrt{dmo\varphi^{-1}/dm}} T_{\varphi_i^{-1}} \ (i = 1,...n)$,

(2) $V_i(\varphi)V_i(\varphi)^* = M_{\chi^{X_i}} \ (i = 1,...n)$,

(3) $\int\limits_X f(\varphi(x))\eta(x)dm(x) = \sum_{i=1}^{n} \int\limits_X \frac{dmo\varphi_i^{-1}}{dm} \eta\left(\varphi_i^{-1}(x)\right) dm$ for η in $L^1(X, m)$

Proposition 2.3. Let φ be a MWnL on a measure space (X.m) and $\{V_i = V_i(\varphi)\}_{i=1}^n$ a fimily isometries associated with φ defined in Definition 2.2. Then it follows that

$\{V_i(\varphi)\}_{i=1}^n$ satisfies condition (X, m) in §1, that is , $\{V_i(\varphi)\}_{i=1}^n$ is a f.i.c. on $L^2(X, m)$.

$\pi\left(\alpha_\varphi(f)\right) = \alpha v(\pi(f))$ for all $L^2(X, m)$.

Proposition 2.3. (2) implies that αv is a *-endomorphism of the von Neumann algebra $\pi L^\infty(X)$ and we denote by A_φ the transpose of the restriction of αv to $\pi L^\infty(X)$, then we have

$$(A_\varphi \eta)(x) = \sum \frac{dmo\varphi_i^{-1}}{dm} \eta\left(\varphi_i^{-1}(x)\right)$$

The transformation A_φ is known as Perron-Frobenius operator on $L^1(X)$

Theorem 2.4

Let φ be a MWnL on a measure space (X.m). Suppose there exist a f.i.c. $\{W_i(\varphi)\}_{i=1}^n$ such that W_1 has eigenvalue 1 with eigenvector e and $\alpha W(T) = \alpha v(T)$ for T in M, where M is a Neumann algebra on H. then for any state ω of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi k}$ where ξk 's are in H_e, it follows that $\lim_{n\to\infty} (\alpha^* v)^n (\omega) = \omega_e$ (Norm topology on M*). Moreover , this implies that $\lim_{n\to\infty} \left\| A_{\varphi}^n (\eta) - |e|^2 \right\|_1 = 0.$ Where $\eta = |\xi|^2$ for ξ in H.

3. EXAMPLES OF MWnL

We give typical and interesting examples of map with m laps. Each number in each example indicates the following.

(1) Measure space (X, m) on which a map is given.

(2) Map φ with m laps on X.

(3) Number n for which φ is a MWnL and the partition $\{X_i\}_{i=1}^n$ of X.

(4) $\{V_i\}_{i=1}^n = \{V_i(\varphi)\}_{i=1}^n$ defined in Definition 2.2.

(4-1) an digenvector e for eigenvalue 1 of W_1 and $ONS(e,V) = \{e_k\}_{k=1}^{\infty}$.

(4-2) $ONS(e,V)$ is complete of not.

(5) $\{W_i\}_{i=1}^n$ such that $\alpha v(T) = \alpha w(T)$ for in von Neuman algebra M on $L^2(X)$

(6) The von Neuman algebra M on which $\alpha v = \alpha w$.

(6-1) an eigenvector e for eigenvalue 1 of W_1 and $ONS(e,V) = \{e_k\}_{k=1}^{\infty}$.

(6-2) $ONS(e,W)$ is complete or not.

(7) Perron-Frobenius operator $A\varphi$.

Example 3.1. (Tent map)

(1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map τ defined by $\tau(x) = 1 - |1 - 2x|$.

(3) n=2 and $X_1 = [0,1/2], X_2 = [1/2,1]$.

(4) $V_1 = \sqrt{2}M_{[0,1/2]}T_\tau, V_1 = \sqrt{2}M_{[1/2,1]}T_\tau$.

(5) $$(W_1, W_2) = (V_1, V_2)\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(6) $M = B(L^2[0,1])$

(6-1) $e(x) = 1(x \in [0,1])$ and $e_1 = e, e_2 = M_{[0,1/2]}e_1 - M_{[1/2,1]}e_1$

(6-2) $ONS(e, W)$ is complete

(7) $A_\tau(\tau)(x) = \left(\eta\left(\frac{x}{2}\right) + \eta\left(1 - \frac{x}{2}\right)\right)$

Example 3.2. (Generalized tent map)

(1) X=[0, 1] and m=Lebesgue measure.

(2) $\varphi = \tau_c, (0 < c < 1)$ defined by

$$\varphi_c(x) = \begin{cases} \frac{1}{c}x & for 0 \le x \le c \\ \frac{1}{c-1}(x-1) & for 0 \le x \le 1 \end{cases}$$

(3) n=2 and $X_1 = [0,1/2], X_2 = [1/2,1]$

(4) $V_1 = M_{\sqrt{1/c}}M_{\chi[0,c]}T_{\lambda c}, V_1 = M_{\sqrt{1/(1-c)}}M_{\chi[c,1]}T_{\lambda c}$

(5) $$(W_1, W_2) = (V_1, V_2)\begin{pmatrix} \sqrt{c} & \sqrt{1-c} \\ -\sqrt{1-c} & -\sqrt{c} \end{pmatrix}$$

(6) $M = B(L^2[0,1])$

(6-1) $e(x)=1(x\in[0,1])$ and $e_1=e, e_2(x)=\begin{cases}\frac{1}{\sqrt{c}} & for 0\le x\le c\\ -\frac{1}{\sqrt{c-1}} & for c<x\le 1\end{cases}$

(6-2) $ONS(e,W)$ is complete.

(7) $A_{\pi}(\tau)(x)=c(\eta(cx))+(1-c)\eta((c-1)x+1)$.

Remark: τ_c and τ_c are topologically conjugate (cf,[6])

Example 3.3 (Logistic map) (cf.[9])

(1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map λ defined by $\lambda(x)=4x(1-x)$

(3) n=2 and $X_1=[0,1/2], X_2=[1/2,1]$

(4) $V_1=\frac{1}{2\sqrt{1-2x}}M_{\chi[0,1/2]}T_{\lambda}, V_2=\frac{1}{2\sqrt{1-2x}}M_{\chi[1/2,1]}T_{\lambda}$.

(5) $(W_1,W_2)=(V_1,V_2)\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$

(6) $M=B(L^2[0,1])$

(6-1) $e_1(x)=e(x)=1/\sqrt{\pi\sqrt{x(1-x)}} and, e_2(x)=M_{\chi[0,1/2]}e_1-M_{\chi[1/2,1]}e_1$

.

(6-2) $ONS(e,W)$ is complete.

(7) $A_{\pi}(\tau)(x)=\frac{1}{4\sqrt{1-x}}\left(\eta\left(\frac{1-\sqrt{1-x}}{2}\right)-\frac{1+\sqrt{1-x}}{2}\right)$.

Remark: The logistic map is topologically conjugate to the tent map with conjugacy $h(x)=\sin^2(\pi x/2)$ (cf.[7]).

Example 3.4. (Typical map with 3 laps)

(1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map λ defined by

$$\varphi(x)=\begin{cases}3x & for 0\le x\le 1/3\\ 3x-1 & for 1/3\le x<2/3\\ 3x-2 & for 2/3\le x\le 1\end{cases}$$

(3) n=3 and $X_1=[0,1/3), X_2=[1/3,2/3), X_3[2/3,1]$.

(4) $V_1=\sqrt{3}M_{\chi[0,1/3]}T_\varphi, V_2=\sqrt{3}M_{\chi[1/3,2/3)}T_\varphi, V_3=\sqrt{3}M_{\chi[2/3,1]}T_\varphi$

(5) $$(W_1,W_2,W_3)=(V_1,V_2,V_3)\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}} & \frac{(3-\sqrt{3})}{6} & \frac{(-3-\sqrt{3})}{6}\\ \frac{1}{\sqrt{3}} & \frac{(-3-\sqrt{3})}{6} & \frac{(3-\sqrt{3})}{6}\end{pmatrix}$$

(6) $M=B(L^2[0,1])$

(6-1) $e(x)=1(x\in[0,1])$ and

$$e_1=e,\ e_1(x)=\chi_{[0,1/3)}+\frac{\sqrt{3}-1}{2}\chi_{[1/3,2/3)}+\frac{-\sqrt{3}-1}{2}\chi_{[2/3,1]}$$

$$e_2(x)=\chi_{[0,1/3)}+\frac{\sqrt{3}-1}{2}\chi_{[1/3,2/3)}+\frac{-\sqrt{3}-1}{2}\chi_{[2/3,1]}$$

(6-2) $ONS(e,W)$ is complete.

(7) $A_\varphi(\eta)(x)=1/3(\eta(x/3))+\eta((x/3+1/3)+\eta((x/3+2/3))$

Example 3.5. (MW2L on [0, 1] such that V_1 has an eigenvector for eigenvalue 1:a)

(1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map defined by

(3) n=2 and $X_1=[0,1/2], X_2=[1/2,1]$

(4) $V_1 = M_{\chi[0,1/4]} + \sqrt{3}M_{\chi[1/4,1/2]}, V_2 = \sqrt{2}M_{x[1/2,1]}$

(4-1)

$e_1 = e,\ e_1(x) = 2\chi_{[0,1/4)}, e_2(x) = 2\sqrt{2}\chi_{(7/8,1]}, e_3(x) = 2\sqrt{6}\chi_{(11/24,1/2]},$

$e_4(x) = 4\chi_{[1/2,9/16)}$

(4-2) $ONS(e,W)$ is complete.

(5) $M = B(L^2[0,1])$

(6) $A_\varphi(\eta)(x) = \eta(x)\chi_{[0,1/4]}(x) + 1/3\eta(\frac{2x+1}{6})\chi_{[1/4,1]} + 1/2\eta(\frac{-x+2}{2})$.**Exa**

mple3.6. (MW2L on [0,1] such that V_1 has an eigenvector for eigenvalue 1:b)

(1) (1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x) = \begin{cases} -5x+1 & for 0 \le x \le 1/8 \\ -x+1/2 & for 1/8 \le x < 1/2 \\ 2x-1 & for 1/2 \le x \le 1 \end{cases}$$

(3) n=2 and $X_1 = [0,1/2], X_2 = [1/2,1]$

(4) $V_1 = \sqrt{5}M_{\chi[0,1/8]} + M_{\chi[1/8,1/2]}, V_2 = \sqrt{2}M_{x[1/2,1]}$

(4-1)

$e_1 = e = 2\chi_{[1/8,3/8]}, e_2(x) = 2\sqrt{2}\chi_{[9/16,11/16]}, e_3(x) = 2\sqrt{10}\chi_{(5/80,7/80]},$

$e_4(x) = 4\chi_{[25/32,27/32]}$

(4-2) $ONS(e,W)$ is complete.

(5) $M = B(L^2[0,1])$

(6)

$$A_{\varphi}(\eta)(x)=\eta(\tfrac{-2x+1}{2})\chi_{[0,1/8]}(x)+1/5\eta(\tfrac{-x+1}{5})\chi_{[1/8,1]}+1/5\eta(\tfrac{-x+1}{2}).$$

Example 3.7. (Square root map)

(1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x)=\begin{cases}\sqrt{2}x & for 0\le x\le 1/2\\ 1-\sqrt{2x-1} & for 1/2\le x\le 1\end{cases}$$

(3) n=2 and $X_1=[0,1/2], X_2=[1/2,1]$

(4) $V_1=(1/\sqrt{2x}M_{\chi[0,1/2]}T_{\varphi}, V_1=(1/\sqrt{2x-1}M_{\chi[1/2,1]}T_{\varphi}$

(5) $(W_1,W_2)=(V_1,V_2)\begin{pmatrix}M_{\sqrt{2cx}} & M_{\sqrt{1-2cx}}\\ M_{\sqrt{1-2cx}} & M_{\sqrt{2cx}}\end{pmatrix}$

(6) $M=M_{L\infty[0,1]}$

(6-1)

$$e_1(x)=e(x)=1, e_2(x)=\sqrt{\left(\tfrac{1}{2x}-1\right)}\chi_{[0,1/2]}(x)-\sqrt{\left(\tfrac{1}{\sqrt{2x-1}}-1\right)}\chi_{[1,1/2]}(x)$$

(6-3) Now we cannot find whether $ONS(e,W)$ is complete or not.

(7) $A_{\varphi}(\eta)(x)=\frac{1}{x}(\eta(\frac{x^2}{2})+\frac{1}{x-1}\eta(\frac{x^2-2x+2}{2}))$

Example 3.8.(map of broken line)

(1) X=[0, 1] and m=Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x)=\begin{cases}\frac{8x}{5} & for 0\le x<1/4\\ \frac{12x-1}{5} & for 1/4\le x<1/2\\ \frac{-12x+13}{7} & for 1/2\le x<17/20\\ \frac{-8x+8}{3} & for 17/20\le x\le 1\end{cases}$$

(3) n=2 and $X_1 = [0,1/2], X_2 = [1/2,1]$

(4) $V_1 = (\sqrt{\tfrac{8}{5}}M_{\chi[0,1/4)} + \sqrt{\tfrac{12}{5}}M_{\chi[1/4,1/2)})T_\varphi$,

$V_2 = (\sqrt{\tfrac{12}{7}}M_{\chi[1/2,17/20)} + \sqrt{\tfrac{8}{3}}M_{\chi[17/20,1]})T_\varphi$

(5) $(W_1, W_2) = (V_1, V_2)\begin{pmatrix} \sqrt{\tfrac{5}{8}}M_{\chi[0,2/5)} + \sqrt{\tfrac{5}{12}}M_{\chi[2/5,1]} & \sqrt{\tfrac{3}{8}}M_{\chi[0,2/5)} + \sqrt{\tfrac{7}{12}}M_{\chi[2/5,1]} \\ \sqrt{\tfrac{3}{8}}M_{\chi[0,2/5)} + \sqrt{\tfrac{7}{12}}M_{\chi[2/5,1]} & \sqrt{\tfrac{5}{8}}M_{\chi[0,2/5)} + \sqrt{\tfrac{5}{12}}M_{\chi[2/5,1]} \end{pmatrix}$

(6) $M = B(L^2\lfloor 0,2/5\rfloor) \oplus B(L^2\lfloor 2/5,1\rfloor)$

(6-1) $e_1(x) = e(x) = 1, e_2(x) = \begin{cases} \sqrt{\tfrac{3}{5}} & for 0 \le x < 1/4 \\ \sqrt{\tfrac{7}{5}} & for 1/4 \le x < 1/2 \\ -\sqrt{\tfrac{7}{5}} & for 1/2 \le x < 17/20 \\ -\sqrt{\tfrac{5}{3}} & for 17/20 \le x \le 1 \end{cases}$

(6-2) Now we cannot find whether $ONS(e,W)$ is complete or not.

(7) $A_\varphi(\eta)(x) = \tfrac{5}{8}\eta(\tfrac{5x}{8})\chi_{[0,2/5]}(x) + 5/12\eta(\tfrac{5x+1}{5})\chi_{[2/5,1]} + 3/8\eta(\tfrac{-3x+8}{8})\chi_{[0,2/5)}(x) +$

Example 3.9.(Product of tent maps)

(1) X=[0, 1]× [0, 1]and m=Lebesgue measure.

(2) φ is the map defined by $\varphi(x,y) = (\tau(x), \tau(y))$, where τ is the tent maps defined in example 3.1.

(3) n=4\and

$X_1 = [0,1/2) \times [0,1/2), X_2 = [1/2,1] \times [0,1/2), X_3 = [1/2,1] \times [1/2,1], X_4 = [0,1/2) \times [1/2,1]$

(4) $V_1 = 2M_{\chi[0,1/2]\times[0,1/2]}T_\varphi, V_2 = 2M_{\chi[1/2,1]\times[0,1/2]}T_\varphi$,

$V_3 = 2M_{\chi[0,1/2]\times[1/2,1]}T_\varphi, V_4 = 2M_{\chi[1/2,1]\times[1/2,1]}T_\varphi$

(5) $$(W_1, W_2, W_3, W_4) = (V_1, V_2, V_3, V_4)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(6) $M = B(L^2\lfloor 0,1\rfloor) \times (L^2\lfloor 0,1\rfloor)$

(6-1) $e_1(x, y) = e(x, y) = 1((x, y) \in [0,1] \times [0,1])$

$e_2(x) = \chi_{[0,1/2]\times[0,1/2]} - \chi_{[1/2,1]\times[0,1/2]} + \chi_{[0,1/2]\times[1/2,1]} - \chi_{[1,1/2]\times[1/2,1]}$

(6-2) $ONS(e, W)$ is complete.

(7) $A_\varphi(\eta)(x) = \frac{1}{4}(\eta(\frac{x}{2}, \frac{x}{2}) + \eta(1 - \frac{x}{2}, \frac{x}{2}) + \eta(\frac{x}{2}, 1 - \frac{x}{2}) + \eta(1 - \frac{x}{2}, 1 - \frac{x}{2}))$

Example 3.10. (Baker's transformation)

(1) X= [0, 1]× [0, 1] and m=Lebesgue measure.

(2) φ is the map defined by

$$\beta(x, y) = \begin{cases} (2x, \frac{y}{2}) & for 0 \le x \le 1/2 \\ (2x - 1, \frac{y+1}{2}) & for 1/2 \le x \le 1 \end{cases}$$

(3) n=1 and $X_1 = X$

(4) $V_1 = T_\beta$

(5) (4-1) $e_1(x) = e(x) = 1$

(4-2) $ONS(e, W) = \{e_1\}$ is not complete.

(6) $M = B(L^2\lfloor 0,1\rfloor) \times (L^2\lfloor 0,1\rfloor)$

(7) $A_\beta(\eta)(x) = \eta(\beta^{-1}(x))$

Remark: Baker's transformation is strong mixing but $\{(\alpha * v)^n(\omega\xi)\}_{n=1}$ does not converges to ω_e in the norm topology in M*.

Example: (Unilateral shift map)

(1) $X = \prod_{n=1}^{\infty}\{1,2\}$ and m=Lebesgue measure.

(2) φ is the map σ defined by

$$\sigma((x_1, x_1, ...,)) = ((x_3, x_4, x_5, ...,)),$$

(3) n=2 and

$$X = X(1) = \left\{(x_n)_{n=1}^{\infty} \in X / x_1 = 1\right\}, X_2 = X(2) = \left\{(x_n)_{n=1}^{\infty} \in X / x_1 = 2\right\}$$

(4) $V_1 = \sqrt{2}M_{x(1)}T_\sigma$, $V_2 = \sqrt{2}M_{x(2)}T_\sigma$

(5) $(W_1, W_2) = (V_1, V_2)\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

(6) $M = B(L^2(X))$

(6-1) $e(x) = 1(x \in X)$ and $e_1 = e, e_2 = k_{x(1)}e_1 - k_{x(2)}e_1$

(6-2) $ONS(e, W)$ is complete.

(7) $A_\sigma(\eta)(x) = \frac{1}{2}\left(\eta\left(\sigma_1^{-1}(x)\right) + \eta\left(\sigma_2^{-1}(x)\right)\right)$ where

$$\sigma_1^{-1}((x_1, x_2, x_3, ...)) = (1, x_1, x_2, ...) and \sigma_2^{-1}((x_1, x_2, x_3, ...)) = (2, x_1, x_2, ...)$$

Example 3.12. (MW2L on the set m of all natural numbers)

(8) X=m and m is counting measure.

(9) φ is the map defined by

$\varphi(2k-1) = k$ And $\varphi(2k) = k(k \in m)$

(3) n=2 and $X_1 = 2m - 1, X_2 = 2m$

(4) $V_1 = \sqrt{2} M_{\chi(2m-1)} T_{\varphi}, V_2 = \sqrt{2} M_{\chi(2m)} T_{\varphi}$

(4-1) $e = e_1 = \chi(1)$ and the sequence $\{e_k\}_{k=1}^{\infty}$ is the canonical CONS of $l^2(m)$.

(4-2) $M = B(l^2(m))$.

(5) $A\varphi(\eta)(k) = \eta(2k-1) + \eta(2k)$.

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