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CHAOTIC CONDITIONS ON COMPACT SPACES AND REDUNDANCY IN THE DEFINITION OF CHAOS

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ABSTRACT

In this article, we show mutual relationship among the chaotic three conditions of the most widely utilized definition of chaos due to R. L. Devaney and in this case we consider compact subspaces of the real line. We also survey the articles by the mathematicians on redundancy in the definitions of chaos by R. L. Devaney and consolidate these definitions with corresponding examples.

Keywords: Redundancy, Compact, Relationship, Chaotic, Definition.

1. INTRODUCTION

Dynamical system is a marvelous new area of mathematics and has received a great deal of attention in recent years. It is an important branch of mathematics of interest to scientists in many disciplines. There has been no universally accepted mathematical definition of chaos, but the most widely utilized definition of chaos is due to R. L. Devaney [Devaney R L, 1989]. He introduces three conditions of chaos (defined as in 3.1). These conditions are known as **(i) density (ii) transitivity (iii) sensitivity.** These conditions can be re-named as **(i) an element of regularity (ii) an irreducibility condition (iii) an element of unpredictability.**

It is very important to note here that density and transitivity are topological properties but sensitivity is that of a metric. It is appropreate for chaos to be topological property and invariant under the action of topological conjugacy.

Chaos explain how very small changes in the initial configuration of a system model may lead great discrepancies over time. The phenomenon called " butterfly effect" accounts for our inability to make accurate prediction in the weather despite enormous computer power and loads of data. This condition captures the idea that in chaotic systems small errors in experimental readings eventually lead to large scale divergence.

The sensitivity condition is famous as "butterfly effect" **(**defined as in 2**)** and it is considered as being the central idea in chaos.

Sensitivity or expansiveness are recognized as the notion of unpredictability and S. MacEachern and L. Berliner in [MacEachern S. and Berliner L. 1993] highlighted these two concepts under the case of a compact set of R, the set of real numbers.

The requirement of density property is less precise than sensitivity and it appeals to those looking for patterns or somewhat regularity within a seeming random system such that density implies that there is order in chaos.

J. Banks et al. [Bank J. *et al.*1992] shown a redundancy namely *'density and transitivity imply sensitivity'. That is,* $(i)+(ii) \Rightarrow (iii)$. This turn means that chaos is a topological property but not the metric properties.

Vellekoop and Berglund [Vellekoop M. and Berglund R. 1994] shown a another redundancy on intervals namely *transitivity is equal to chaos*.

Crannell mentioned that the result by Banks et al. yielded a less intuitive definition of chaos and asked 'Why transitivity and why not something else' ? [Crannell A. 1995]. In his paper entitled *The Role of Transitivity in Devaney's Definition of Chaos*, he suggested that a slightly more natural concept **blending** as an alternative to transitivity.

After that Touhey [Touhey P .1997] introduced another definition of chaos which is equivalent to that of Devney's . Research is going on the definition of chaos and we will continue this survey.

2. CHAOTIC PHENOMENA, SENSITIVITY AND BUTTERFLY EFFECT

Chaotic Phenomena: Much of what happens in nature can be modeled by mathematical equations. For years, scientists have been developing mathematical models from the motion of a simple pendulum to the motion of a planet in the solar systems. There are equations for the rise of fall of populations and ups of down of the economy.

Now in many cases, the mathematical model involved is itself very difficult to solve exactly. Most often scientists are using computer to get approximate solutions of mathematical models. Unfortunately, despite of major improvements in computational speed and accuracy, Scientists have often been unable to make prediction based on the output of the computer. For example, it is impossible to accurately forecast the weather one week ahead of time though we can know the current weather at virtually every point on the globe at any given moment.

For years scientist thought that they would be able to make accurate predictions if they could have access to bigger and better mechanics on to faster algorithm or to more initial data. However, in the last 25 years, scientists and mathematicians have come to realize that, that can never be the case. The culprit is the mathematical phenomenon known as chaos. When chaos is a part of mathematical model, faster and more accurate computing will never lead to complete predictability.

Sensitivity: If we look at the logistic iteration rule $x \to 4x(1-x)$ with two nearby seeds (0.5 and 0.5001) the orbit of seed 0.5 is closed to that of seed 0.5001 for fast 13 or 50 iteration. After that they move away very differently. This phenomenon is called sensitivity to initial conditions.

Sensitivity to initial condition means that the orbits of to nearby seeds behave very differently after some iteration.

Example of Sensitivity: Does pendulum exhibit sensitivity to initial conditions? No, but what happens if we drop a pencil point first towards the ground ? Does the final resting place change if we drop the pencil again ? Some physical system are sensitive, some are insensitive.

Butterfly Effect: If a butterfly flaps its wing in a distance place such as Japan, then it will not have a tremendous effect on the weather in Japan right away. But the ripples in the air caused by slight flapping of wings change the various wind speeds and directions (ever so slightly at first), this could set in motion that radically after the worlds weather sometimes much latter. This is real sensitivity to initial conditions.

3. RELATIONSHIP AMONG THE CHAOTIC THREE CONDITIONS

3.1 Definition by R. L. Devaney (1989)

Let X be a meric space. A continuous function $f : X \to X$ is said to be *chaotic(Dev.)* on X if f has the following three properties:

(C-1) Periodic points are dense in the space *X*

(C-2) f is topologically transitive

(C-3) f has sensitive dependence on initial conditions Mathematically,

(C-1)Per(f)= $\{x \in X : f^k(x) = x(\exists k \in N)\}$ is dense in X.

(C-2) For $\forall\, U, V : non-emptyopensets$ of X, $\exists k \in N$ such that $f^k(U) \cap V \neq \phi$.

(C-3) $\exists \delta > 0$ (sensitive constant) which satisfies:

$\forall\ x \in X\ and\ \forall N(x,\varepsilon),\ \exists y \in N(x,\varepsilon)$ and $\exists\ k \leq 0$ such that $d(f^k(x), f^k(y)) > \delta.$

3.2. Examples of chaotic map by Devaney

The infinite product of two points is called the Cantor set and in this paper we use two kinds of Cantor sets with metrics, namely

$$\sum_+ = \{(X_n)_{n\in \mathrm{N}} : X_n \in \{0,1\}\} \text{ with metric } d_+,$$

where $d_+(x,y) = \sum_{n=1}^{\alpha} \frac{|x_n - y_n|}{2^n}$ for $x = (x_n)_{n\in \mathrm{N}}$ and $y = (y_n)_{n\in \mathrm{N}}$ in $\sum_+$. And $\sum = \{(x_n)_{n\in Z} : x_n \in \{0,1\}\}$ with metric d, where

$d(x,y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n} + \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$ for $x = (x_n)_{n\in Z}$ and $y = (y_n)_{n\in Z}$ in Σ.

Example-1 (Shift Map). Let $X = \Sigma_+$ and S_+ be the map of X onto X defined by $y = S_+(x)$ where $x = (x_n)_{n\in Z}$, $y = (y_n)_{n\in Z} \in \Sigma_+$ and $y_n = x_{n+1}$ for all n in N.

Example-2(Shift Map). Let $X = \Sigma$ and S be the map of X onto X defined by $y = S(x)$ where $x = (x_n)_{n\in Z} \in \Sigma$, $y = (y_n)_{n\in Z} \in \Sigma$ and $y_n = x_{n+1}$ for all n in Z.

Example-3 (Logistic Map). Let $X = [0,1]$, $f : X \to X$ defined by $f(x) = 4x(1-x).$

Example-4 (Tent Map). Let $X = [0,1]$, $f : X \to X$ defined by $f(x) = 1 - |1-2x|.$

We can show that all these map satisfies the Devaneys chaotic conditions (C-1), (C-2) and (C-3).

Example -5. Let $X = (1, \alpha)$, with $d(x, y) = |x - y|$ and f be the continuous function defined by $f(x) = 2x$.

Example-6. Let $Y = (0, \alpha)$, with $d(x, y) = |x - y|$ and g be the continuous function defined by $g(x) = x + \log 2$.

Proposition: Let *X*, *Y* and *f*, *g* be as in Example-4 and Example-5. Then we have the following:

(1) $h(x) = \log x$ is a topological conjugacy between $f(x)$ and $g(x)$

(2) *f* does exhibit sensitive dependence on initial conditions

(3) *g* does not exhibit sensitive dependence on initial conditions

(i.e, (C-3) is not preserved by topological conjugacy).

Proof: (1) Since $h(x) = \log x$, thus $h^{-1}(x) = e^x$. We put $g = h \circ f \circ h^{-1}$. Then *f* and *g* are topologically conjugate since

$(h \circ f \circ h^{-1})(x) = h \circ f(e^x) = h(2e^x) = \log 2e^x = \log 2 + \log e^x = x + \log 2$

(2) We put $\delta = 1$. Let $x \in X$ and $\varepsilon > 0$ are given and let $y \in U(x, \varepsilon)$ and $y \neq x$. Then $d(f^k(x), f^k(y)) = |f^k(x) - f^k(y)| = |2^k x - 2^k y| = 2^k |x - y| > 1$ for some $k \in N$. Therefore *f* does not exhibit sensitive dependence on initial conditions.

(3) Let $\delta > 0$ be given and *x* be an arbitrary point of *Y*. we put $\varepsilon = \delta$ and let $y \in U(x, \varepsilon)$. Then we have

$$d(g^k(x), g^k(y)) = \left|g^k(x) - g^k(y)\right|$$

$$= \left|x + k\log 2 - (y + k\log 2)\right|$$
$$= \left|x - y\right| = d(x, y) < \varepsilon = \delta$$

Therefore g does not exhibit sensitive dependence on initial conditions.

In the following we show some results for some maps on compact subspaces which make relationship among the above chaotic three conditions.

3.3. Lemma-1

Let $f:[0,1] \to [0,1]$ be a continuous map. If f is a homeomorphism, then f is not one sided topologically transitive, i.e, f saisfies (C-2).

Proof: Suppose that f is a homeomorphism. Then f is monotonically increasing or monotonically decreasing. For each case, we show that f is not one sided topologically transitive. First we consider the case of monotonically increasing, in which $f(0) = 0$ and $f(1) = 1$. Let x_0 be a point in the open interval $(0,1)$. Then we have the following two cases.

Case (1-1) [$f(x_0) = x_0$] Let $U = (0, x_0)$ and $V = (x_0, 1)$. Then we have $f^n(U) \cap V = U \cap V = \phi (n \in N)$.

Case (1-2) [$f(x_0) \neq x_0$] Let $a = \min\{x_0, f(x_0)\}$, $b = \max\{x_0, f(x_0)\}$, $U = (a, b)$ and $V = f(U)$.

Then we have $f^n(U) \cap V = U \cap V = \phi (n \in N)$.

Next we consider the case of monotonically decreasing, in which $f(0)=1$ and $f(1)=0$. Let y_0 be a point on the open interval $(0,1)$ such that $f(x_0)\neq x_0$. Then we have the following two cases, too.

Case (2-1) [$f^2(y_0)=y_0$]

Let $U=(0,y_0)\cup(f(y_0),1)$ and $V=(y_0,f(y_0))$.

Then we have $f^n(U)\cap V=U\cap V=\phi(n\in N)$.

Case (2-2)[$f^2(y_0)\neq y_0$]

Let $a=\min\{y_0,f^2(y_0)\}$, $b=\max\{y_0,f^2(y_0)\}$, $U=(a,b)$ and $V=f(U)$. Then we have $f^n(U)\cap V=\phi(n\in N)$.

This means that no homeomorphisms of $[a,b]$ are one sided topologically transitive.

3.4. Lemma-2

Let X be compact subspace of **R** which has a non-empty open interval (a,b) and $f:X\to X$ a continuous map. If f is a homeomorphism, then f is not one sided topologically transitive, i.e., *f* does not satisfy (C-2).

Proof: Suppose that f is a homeomorphism and let (c,d) be the largest open interval in X such that (c,d) contains (a,b). Since X is closed in **R**, the closed interval $[c,d]$ is contained in X. In the case where $f^n([c,d])\cap[c,d]=\phi(n\in N)$, trivially f is not one sided topologically transitive. Now we consider the case where $f^n([c,d])\cap[c,d]\neq\phi$ for some $n\in N$. Let k be the smallest positive integer such that $f^k([c,d])\cap[c,d]\neq\phi$. Since $[c,d]$ is a connected component in X

and f^k is a homeomorphism of X onto itself, the set $f^k([c,d])$ is also a connected component in X. Thus we have $f^k([c,d]) = [c,d]$ and the restriction of f^k to $[c,d]$ becomes a homeomorphism of $[c,d]$. Hence by Lemma 3.3, f^k is not one sided topologically transitive on [c, d], that is, there exist two non-empty open intervals U and V in $[c,d]$ such that $f^{kn}(U) \cap V = \phi$ for all $n \in N$. By considering open intervals in U and V, we can take U and V as non-empty open sets in X and we have

$$f^{kn+i}(U) \cap V \subset f^i([c,d]) \cap [c,d] = \phi \qquad \text{for } i = 1,2,\ldots,k-1.$$

Therefore f is not a one sided topologically transitive.

3.5. Lemma-3

Let X be a compact subspace of R which has an isolated point and $f : X \to X$ be a continuous map. Then f does not have sensitive dependence on initial conditions, i.e., *f* does not satisfy (C-3).

Proof: Let x an isolated point. Then there exist $\varepsilon > 0$ such that
$\{y \in X : d(y,x) < \varepsilon\} \cap X = \{x\}.$

Hence for this ε, the condition $d(y,x) < \varepsilon$ implies $y = x$. Thus we have $d(f^n(x), f^n(y)) = 0$ for all $n \in N$.

3.6. Lemma-4

The two sided shift map S is chaotic homeomorphism.

Proof: It is easy to show that S satisfies the three conditions of chaos and also it is bijective and bicontinuous. Consequently, S is chaotic homeomorphism. Combining above Lemmas, we get the following proposition.

3.7. Proposition:

For a compact subspace X of **R**, there exists a homeomorphism of X which satisfies two conditions (C-2) and (C-3) if and only if X is homemorphic to the Cantor set C. Moreover Lemma-4 means that this is equivalent to that there exists a chaotic homeomorphism of X. Namely we have the following theorem

3.8. Theorem([P. Ahmed, S. Kawamura , S. Sasaki]

Let X be a compact subspace of **R**. Then there exists a chaotic homeomorphism $f : X \to X$ if and only if X is homeomorphic to a Cantor set $\sum$.

We note that, in the case where X is a finite subset of **R**, there exists a homeomorphism of X which satisfies two conditions (C-1) and (C-2) but does not satisfy the condition (C-3).

In [Ahmed P. and Kawamura S. 2008], it is shown that chaotic map is a map in a family of chaotic homeomorphism. For a subset M of Z, let $C_M = \prod_{(m,n)\in M\times Z}\{0,1\} = \{x = (x_{(m,n)})_{(m,n)\in M\times Z} : x_{(m,n)} \in \{0,1\}\}$,where C_M has the canonical product topology and S_M be the homeomorphism of C_M onto itself defined by $y = S_M(x)$, where $x = (x_{(m,n)})_{(m,n)\in M\times Z}$, $y = (y_{(m,n)})_{(m,n)\in M\times Z}$ and $y_{(m,n)} = x_{(m,n+1)}$

For each $(m,n)\in M\times Z$. It is easy to show that C_M is homeomorphic to $\sum$ and S_M is a homeomorphism. Then $S_M : C_M \to C_M$ satisfies (C-1), (C-2) and (C-3) and consequently, the following result is found in [ ].

3.9. Lemma

The map $S_M : C_M \to C_M$ is a chaotic homeomorphism.

Let $\varphi : Z \to Z$ be a bijective map and $S_\varphi : \sum \to \sum$ be the homeomorphism defined by $y = S_\varphi(x)$, where $x = (x_n)_{n\in Z}$, $y = (y_n)_{n\in Z}$ and $y_n = x_{\varphi(n)}$ for all n in Z. For the map φ, $O(n) = \{\varphi^i(n)\}_{i\in Z}$. Then following $S_\varphi : \sum \to \sum$ satisfies (C-1), (C-2) and (C-3) and also is homeomorphism. Consequently, the following result is also found in [ ].

3.10. Theorem

$S_\varphi : \sum \to \sum$ is a chaotic homeomorphism if and only if $O(n)$ is an infinite set for all n in Z.

4. REDUNDANCY IN THE DEFINITIONS OF CHAOS AND CORRESPONDING EXAMPLES

4.1. Definition by J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey (1992)

J. Banks et al. [Bank J. *et al.*1992] mentioned the following redundancy in the Devaney's definition of chaos:

4.1.1. Theorem

Let X be a meric space and f be a continuous function on X. If f is topologically transitive and has dense periodic points, then f has sensitive dependence on initial conditions.

Namely, transitivity and density of periodic points imply sensitivity.

Needless to say , though the conditions (C-1) and (C-2) are topological properties, the condition (C-3) is not a topological property but metric one. If *X* is compact metric space. The condition (C-3) also becomes a topological property which is shown in [Bank J. *et al.*1992].

Thus the definition of chaotic map can be re-written as follows:

(C-1) The set of all periodic points is dense in X.

(C-2) for any pair of non-empty open sets U and V in X there exists $k \in N$ such that $f^k(U) \cap V \neq \phi$

If we droop the condition that *X* has infinitely many points then the Theorem 4.1.1 does not hold. The following is a counter example.

Example. Let $X = \{x_1, x_2, x_3, ..., x_n\}$ with $d(x_i, x_j) = \begin{cases} 1 if i \neq j \\ 0 if i = j \end{cases}$ and *f* be the map defined by $f(x_i) = x_{i+1}$, $f(x_n) = x_1$ for $1 \le i \le n-1$.

4.1.2 Proposition.

Let X and *f* be as in just above Example, then *f* satisfies (C-1) and (C-2).

Proof: We have $Per(f) = P_n(f) = X$, where *Per(f)* and *$P_n(f)$* are the sets of all periodic and *n* periodic points. Thus $Per(f)$ is dense in X. Therefore *f* satisfies (C-1). Now let U and V be two non-empty open sets in X. Thus there exist $x_i \in U$ and $x_j \in V$ .When $i \le j$, we put $k = j - i$. Then $f^k(x_i) = x_{i+k} = x_j$. Thus $f^k(x_i) = x_j \in f^k(U) \cap V \neq \phi$ .Hence *f* satisfies the condition (C-2). When $i > j$, we put $k = n - i + j$.

Thus $f^k(x_i) = f^{n-i+j}(x_i) = f^l(f^{n-i}(x_i)) = f^j(x_n) = x_j$.

Thus $f^k(U) \cap V \neq \phi.$

Therefore *f* satisfies the condition (C-2).

In this case $n = 2$,

$X = \{a, b\}(a \neq b), d(a,a) = d(b,b) = 0, d(a,b) = 1,$
$O(d) = \{\varphi, \{a\}, \{b\}, X\}$. The map $f : X \to X$ defined by $f(a) = b$,

$f(b)=a$, $f^2(a)=a$, $f^2(b)=b$ .Then $Per(f)=P_2(f)=\{a,b\}=X$. Thus $per(f)$ is dense in X. Namely (C-1) holds. Next let U and V are non-empty open sets. Then (1) $a\in U$ and $a\in V$ then $f^0(U)\cap V\neq\phi$.

(2) $a\in U$ and $b\in V$ then $f(U)\cap V\neq\phi$

(3) $b\in U$ and $a\in V$ then $f(U)\cap V\neq\phi$

(4) $b\in U$ and $b\in V$ then $f^0(U)\cap V\neq\phi$. Thus f does not satisfy (C-2).

4.2 Definition by Assaf IV and Gadbois (1992)

In the same volume, Assaf IV and Gadbois [AG 1992] shown that this is only redundancy for a general map**:**

4.2.1 *Let X be a meric space and f be a continuous function on X. (1) If f is topologically transitive and has sensitive dependence on initial conditions, then f has no dense periodic points, i.e.,* $(ii)+(iii)$ *do not imply* (i) *(2) If f has dense periodic points and has sensitive dependence on initial conditions, then f is not topologically transitive, i.e.,* $(i)+(iii)$ *do not imply* (ii).

4.2.1.1 Example for (1): Let $X=S^1\setminus\{\exp(i2\pi x/y):x,y\in Z\}$ and $f(\exp(i\theta)=\exp(i2\theta)$. Then the set of periodic points of f is empty. This means, density condition does not hold.

4.2.1.2 Example for (2): Let X be a cylinder $S^1\times[0,1]$ with the induced "taxicab" metric and $f(\exp(i\theta,t)=(\exp(i2\theta),t)$. Taking $U=S^1\times[0,\frac{1}{2})$ and $V=S^1\times[\frac{1}{2},1]$) ,then U cannot intersect with V under the iterations of f. This means, f is not topologically transitive.

4.2.1.3 Example for (2): Let $X = \Re^1$ and f be defined by

$$f(x) = \begin{cases} 3x, x \in [0, \frac{1}{3}) \\ -3x + 2, x \in [\frac{1}{3}, \frac{2}{3}) \\ 3x - 2, x \in [\frac{2}{3}, 1) \\ f(x-1) + 1, x \in [1,2]. \end{cases}$$

Then *f* has sensitivity because of its expansiveness $\left|\frac{df}{dx}\right| = 3$. Taking U=[0,1) and V=(1,2] , then U cannot intersect with V under the iterations of *f*. This means, f is not topologically transitive.

4.3 Definition by M. Vellekoop and R. Berglund [1994]:

Here attention is restricted to maps on an interval and the following result is obtained by M. Vellekoop and R. Berglund.

4.3.1 ***Proposition:*** *Let I be an interval (not necessarily finite) and* $f : I \to I$ *be a continuous function. If f is topologically transitive, then f has density and sensitivity, i.e.,* $(ii) \Rightarrow (i) + (iii)$. *This means, on an interval, transitivity = chaos.*

Namely both sensitivity and density are redundant conditions in the definition of chaos for maps on an interval. Also they note that there are no other trivialities in the Devaney's definition of chaos when attention is restricted to intervals. On intervals, neither density nor sensitivity is enough to ensure any of the other conditions of chaos.

We have already given an example [**4.2.1.3 Example for (2)]** on an interval showing that density and sensitivity do not imply transitivity.

The following is an example [Vellekoop M and Berglund R .1994] on an interval satisfying that sensitivity does not imply density:

4.3.1.1 Example: Let $I = \left[0, \frac{3}{4}\right]$ and f be defined as

$$f(x) = \begin{cases} \frac{3}{2}x, if\ 0 \le x < \frac{1}{2}; \\ \frac{3}{2}(1-x), if\ \frac{1}{2} \le x \le \frac{3}{4}. \end{cases}$$

Also they note an identity map on an interval as an example satisfying that density does not imply sensitivity. In this case, we should be very much interested for a non-linear example on an interval.

4.4 Definition by A. Crannell (1995)

M. Vellekoop and R. Berglund studied onle the transitivity for one dimensional dynamical system. In 1995, A. Crannell suggested a concept **blending** as an alternative of transitivity.

Let $M \subset \Re^n$ *and f be continuous on M. Then f is* ***weakly blending*** *if,* $\forall U, V(\neq \phi) \in MM \subset \Re^n, \exists K > 0$ *such that* $f^k(U) \cap f^k(V) \neq \phi,$ *f is* ***strongly blending*** *if,* $\forall U, V(\neq \phi) \in MM \subset \Re^n, \exists K > 0$ *such that* $f^k(U) \cap f^k(V)$ *contains a non-empty open subset.*

The following are the examples to show that blending functions are not necessarily transitive, and transitive functions are not necessarily blending:

4.4.1 Example: Let $X = S^1$ and $f : S^1 \to S^1$, given by $f(\theta) = \theta + k$, where $\frac{k}{\pi}$ is irrational. Then f is transitive but not strongly or weakly blending.

4.4.2 Example: Let X = [-1, 1] and $f : X \to X$ satisfying (i) $|f'(x)| > 2,$ on except at the vertices of f (ii) each vertex of the graph of f likes alternately on the line $y = \frac{x}{2}$ and $y = -\frac{x}{2}$. Then f is strongly blending but not transitive.

The following is an example on an interval to show that weakly blending does not imply transitivity and nor strongly blending:

4.4.3 Example: Let I = [-1, 1] and f be defined as

$$f(x) = \begin{cases} -2x-2, if\ -1 \le x \le -\tfrac{1}{2} \\ 2x, if\ -\tfrac{1}{2} \le x \le \tfrac{1}{2} \\ 2-2x, if\ \dfrac{1}{2} \le x \le 1. \end{cases}$$

As remark A. Crannell mentioned that the Examples in 4.4.2 and 4.4.3 do not satisfy density property. Hence adding density condition to blending conditions, she obtains the following theorem:

4.4.4 Theorem 1[**A. Crannell 1995;** ***Density and strongly bending condition imply chaos***]**:**

Let M be a subset of $\Re^n$ *and f be a continuous function on M with dense periodic points. Then if f is strongly blending, f is also transitive.*

This means, density and strongly bending Acondition imply chaos. If *M* is restricted to a compact set in $\Re^1$, she gives another theorem**:**

4.4.5 Theorem 2[**A. Crannell 1995**]

Let M be a compact subset of $\Re^1$ *and f be a continuous function on M with transitivity and a reppeling fixed point. Then f is weakly blending.*

4.5 Definition by P. Touhey(1997)

P. Touhey introduces another definition of chaos which is equivalent to Devaney's one. He gave a simple and concise definition of chaos reformulating the two topological conditions of transitivity and density of periodic points as a single condition.

Definition: Let *X* be a merit space. A continuous function $f : X \to X$ is said to be *chaotic(Tou.)*

if, $\forall U, V : non-empty open sets of X, \exists\ p(periodic point) \in U$ and $k \geq 0$ such that $f^k(p) \in V$.

This means, every pair of non-empty open subsets of *X* shares a periodic point.

4.5.1Theorem [**P. Touhey1997]**

Let X be a merit space and f be a continuous function on X. Then the are:(i) f is chaotic(Tou.)(ii) f is chaotic(Dev.) (iii) any finite collection of non-empty open subsets of X shares a periodic point and (iv) any finite collection of non-empty open subsets of X shares infinitely many periodic orbits are equivalent.

4.5.2 On the definition of 'chaos [Andrei Yu Kolesov and Nikolai Kh Rozov 2009]:

A new definition of a chaotic invariant set is given for a continuous semiflow in a metric space. It generalizes the well-known definition due to Devaney and allows one to take into account a special feature occurring in the non-compact infinite-dimensional case: so-called turbulent chaos. The paper consists of two sections. The first contains several well-known facts from chaotic dynamics, together with new definitions and results. The second presents a concrete example demonstrating that their definition of chaos is meaningful. Namely, an infinite-dimensional system of ordinary differential equations is investigated having an attractor that is chaotic in the sense of the new definition but not in the sense of Devaney or Knudsen.

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