COLLABORATIVE MANIPULATORS HANDLING FLEXIBLE OBJECT IN JOINT SPACE-
MODELING, CONTROL AND ANALYSIS

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ABSTRACT

This paper focus on the dynamic modeling and nonlinear robust control of two planar rigid manipulators moving a flexible object from any initial position and orientation (pose) to the desired pose while suppressing the vibration of the flexible object. Dynamic equations of the flexible object are derived using Hamiltonian principle, which is expressed as a partial differential equation (PDE) with appropriate boundary conditions. Then, a combined dynamics is formulated by combining the manipulators and object dynamics without any approximation in joint space. The resulting dynamics are thus described by the PDE’s, having rigid as well as flexible parameters coupled together. This paper attempts to develop a robust control scheme without approximating the PDE in order to avoid measurements of flexible coordinates and their time derivatives. For this purpose, the two subsystems, namely, slow and fast subsystems, are identified by using singular perturbation technique. Specific robust controllers for both the subsystems are developed. In general, usage of the singular perturbation technique necessitates exponential stability of both subsystems, which is evaluated by satisfying the Tikhnov’s theorem. Hence, the exponential stability analysis is performed for both subsystems. Focusing on two three link manipulators holding a flexible beam, simulations are performed and simulation results demonstrate the versatility of the proposed robust composite control scheme.

KEYWORDS: Robotic Manipulator, Singular Perturbations, Slow Subsystem, Fast Subsystem, Composite Control and Exponential Stability

INTRODUCTION

In the past, a single robot alone was not able to grasp and move a long object in a safe and efficient way. Owing to the single arm structure, present day robots are called “handicapped operators” for performing complex tasks. Most tasks in assembly / disassembly, handling large or heavy objects are done efficiently with two robot arms. Collaborative manipulators have the following advantages compared to single arm manipulators: (i) increased load carrying capacity by sharing the loads between the manipulators (ii) greater dexterity and manipulability in handling flexible objects (iii) reduced need for extra auxiliary equipments (iv) efficient use of available workspace (v) increased productivity by operating each robot in parallel to achieve different tasks at the same time. Two robots handling a rigid object have been studied by many researchers for example [1] - [5], whereas, manipulating a flexible object was studied few researchers [6] - [10]. A modern automobile body assembly has more than 200 sheet metal parts which must be assembled in a precise way. Handling them needs special equipment and skilled operation. Two robots can grasp a flexible sheet metal and force them together for assembly. Also, in industry, many deformable objects such as rubber tubes, sheet metals, cords and leather products are handled by special equipment or human operators. The dynamics and control problem of two
manipulators collaboratively handling flexible materials is complex compared with handling the rigid parts. In order to develop an efficient control algorithm a precise dynamic model is necessary and the un precise models may create problems such as control and observation spill over [11]. Many researchers approximated the flexible object dynamic model using Finite Element Method (FEM) and Assumed Mode Method (AMM). The truncation of the original model with infinite degrees of freedom of a flexible beam to a finite dimensional model poses the following issues [12]: 1) Requirement of as many sensors as the locations of the measurement of vibration and the difficulty in implementation. 2) Presence of control and observation spill over due to the ignored high frequency dynamics. 3) It is often not clear how many modes must be considered to approximate the PDE based model into ordinary differential equation (ODE) model. 4) Destabilization of the system due to the negligence of the higher order modes. 5) Necessity of higher order controller. Considering the aforementioned reasons and unlike in the earlier available studies, this paper concerns with an overall objective of development of a dynamic model of manipulators-flexible object system without using any approximation methods and design of a robust control scheme. The purpose of the robust control system design is to use the two planar three link manipulators to move the flexible object in the prescribed position and orientation and simultaneously to suppress the vibration of the flexible object with unknown manipulator and beam parameters.

In order to control the manipulators-flexible object system without using any approximation methods, a possible control approach is the two-time scale theory [13] which considers the high frequency phenomenon of flexible motion in different time scales. The basic idea for the two-time scale theory is to identify the slow and fast subsystems in separate time scales. Then, one can design a control algorithm for each subsystem and they together form a composite control input to achieve the desired objective. Therefore, the two subsystems, namely, slow system describing the rigid body motion and fast subsystems describing the transverse vibration are identified by using singular perturbation technique. Focusing on these two subsystems, specific controllers are developed. The use of singular perturbation technique necessitates the exponential stability of each subsystem which satisfies the Tikhonov’s theorem [14] for the closed loop stability of the system and therefore they are presented. The proposed robust controller has the following features: 1) it does not need the information of the modes; 2) exact knowledge of the manipulators parameters or the beam is not required; and 3) it satisfies the Tikhonov’s theorem.

The remaining content of the paper is organized as follows. Section II presents the kinematics and dynamics of beam without approximating or discretizing the beam. Section III deals with manipulators dynamics and combined dynamics resulting from the amalgamation of beam and manipulators dynamics. Singularly perturbed complete system of dynamic model has been developed in section IV. Then, by incorporating typical steps of singular perturbation approach, the slow and fast subsystems are derived in section V. Therefore, one can design a controller for each subsystem. In Section VI composite control strategy and corresponding stability analysis is carried out. For the control of slow subsystem, a regressor based sliding mode control algorithm is developed. The main advantages of the proposed robust control scheme is that, it can handle quickly varying parameters and alleviate the problem of choosing the upper bounds of the uncertainties in the robust approaches; it also does not need persistency of excitation and guarantees the exponential convergence of transient behavior. In the case of control of fast subsystem, a dissipator operator forms the closed loop system and it damps out the vibration of flexible object. Exponential stability analysis of slow and fast subsystem results validate the composite control strategy with the help of Tikhonov’s theorem. Simulations are performed by considering two three link manipulators rigidly grasping and moving a flexible beam in Section VII. Simulation results demonstrate that the proposed control approach can achieve satisfactory tracking performance while suppressing the vibration of the flexible object.
MATHMATICAL MODELING OF FLEXIBLE OBJECT

In the manufacturing and automobile industries, many components to be assembled can be modeled as beams. Various applications such as, turbine rotor blades, spacecrafts with flexible appendages, flexible robot arms and aerospace systems, are essentially beams which are flexible bodies. In the following analysis, flexible object will be modeled as a beam and in particular Euler-Bernoulli beam theory will be adopted. Since the two manipulators are used to move the object in the desired position and orientation which necessitates the allowance of rotation at the two ends of the beam, the simply supported end conditions to the flexible beam are therefore considered for deriving the dynamic equations of motion of the beam. Certainly, other end boundary conditions can be included for the derivations of the beam dynamics. However, the purpose of this paper is to illustrate the essential features of the controller designs avoiding confusion of the mathematical expressions.

Consider a beam shown in Fig.1 of length L, mass \( m = \rho L \), where \( \rho \) is mass per unit length. The mass center position and orientation of the beam with respect to \( X_1Y_1 \)-frame are given by \( c_0 = (x_0 \ y_0 \ \theta)^T \). \( F_{1x} \), \( F_{1y} \), \( F_{2x} \), \( F_{2y} \) are the forces applied by the manipulator at the two ends of the beam. The transverse displacement \( \eta(x, t) \) is measured with respect to \( XY \)-frame and deformation in the longitudinal direction is neglected. For simplicity, argument \((x, t)\) will be omitted further.

Any point on the beam can be written as,

\[
X = x_0 + x \cos \theta - \eta \sin \theta
\]

(1)

\[
Y = y_0 + y \sin \theta - \eta \cos \theta
\]

(2)

The left end pose (Position and Orientation) of the beam is given by,

\[
\{c_1\} = \{c_0\} - \left( \begin{array}{c}
\frac{\eta}{2} \\
\frac{\eta \sin \theta}{2} \\
0
\end{array} \right) + \mathbf{F}_{1y} \eta \sin \theta \eta \cos \theta
\]

(3)

The right end pose of the beam is given by,

\[
\{c_2\} = \{c_0\} + \left( \begin{array}{c}
\frac{\eta}{2} \\
\frac{\eta \sin \theta}{2} \\
0
\end{array} \right) + \left( \begin{array}{c}
\mathbf{F}_{1y} \eta \sin \theta \eta \cos \theta \\
\mathbf{F}_{2y} \eta \sin \theta \eta \cos \theta \\
0
\end{array} \right)^T
\]

(4)

Figure 1: Flexible Beam with its Rigid Body and Flexible Motion
Differentiating (3) and (4) results

\[
\mathbf{\dot{q}} = \mathbf{[R]} \mathbf{\dot{\theta}}
\]  

The above relation for end-effectors velocity can be written in compact form as,

\[
\mathbf{\dot{q}} = [\mathbf{R}] \mathbf{\dot{\theta}}
\]  

Differentiating (5) gives the end-effectors acceleration

\[
\mathbf{\ddot{q}} = [\mathbf{R}] [\mathbf{\ddot{\theta}}] + [\mathbf{R}] [\mathbf{\dot{\theta}}]
\]  

In order to obtain the dynamical relations for the flexible object, the kinetic energy, potential energy due to elasticity and gravity will be obtained. Further, by using Hamilton’s Principle [15], the dynamic equation of flexible object will be derived.

Kinetic energy of the beam is defined as,

\[
K.E = \frac{1}{2} \int \frac{L}{2} \rho (x^2 + y^2) dx
\]  

Differentiating (1) and (2) gives,

\[
X = x_0 - [x_0 \sin \theta + y_0 \cos \theta + \eta \cos \theta - \eta \sin \theta]
\]

\[
Y = y_0 - [x_0 \cos \theta + y_0 \sin \theta + \eta \sin \theta + \eta \cos \theta]
\]

Squaring (8) and (9) and substitute into (7) yields,

\[
K.E = \frac{1}{2} \int \frac{L}{2} \rho \left[ x^2 + y^2 + \theta^2 \eta^4 + (x \theta + \eta)^2 - 2 \theta \eta (x_0 \cos \theta + y_0 \sin \theta) + 2 (\theta + \eta) (y_0 \cos \theta - x_0 \sin \theta) \right] dx
\]  

Potential energy due to elasticity of the beam is given by,

\[
U_e = \frac{1}{2} \int \sigma_{xx} \varepsilon_{xx} dV
\]

\[
U_e = \frac{1}{2} \int \frac{L}{2} [\sigma \eta^2] dx
\]  

Where, \( \sigma_{xx} = E \varepsilon_{xx} \) \( \varepsilon_{xx} = - \eta \), \( dV = dxdydz \)

Potential energy due to gravity is,

\[
U_g = \rho g \frac{L}{2} (y_0 + x_0 \sin \theta) dx = mgy_0
\]  

Total Potential energy of the beam can be obtained as,

\[
P.E = U_e + U_g
\]
Work done due to the external forces is formulated as

$$W = \left( F_{x_0} (x_0 - \frac{1}{6} \omega_0 + \theta_0) + F_{y_0} (y_0 - \omega \phi_0) + F_{z_0} (z_0 + \omega \phi_0) + \right)$$

(14)

Using Hamilton’s principle,

$$\int_0^L (\delta K - \delta F + \delta W) \, dt = 0$$

(15)

The dynamic equations of rigid body motion along X, Y directions and rotation about Z are derived. These equations are written in compact form as,

$$M_{ref} \ddot{q}_{ref} + C_{ref} + \gamma_{ref} + B_{ref} = F_{ref} (-u)$$

(16)

Where,

$$M_{ref} = \begin{bmatrix} m & 0 & -\rho \cos \theta \int_{\frac{L}{2}}^{L} \eta \, dx \\ 0 & m & -\rho \sin \theta \int_{\frac{L}{2}}^{L} \eta \, dx \\ -\rho \cos \theta \int_{\frac{L}{2}}^{L} \eta \, dx & -\rho \sin \theta \int_{\frac{L}{2}}^{L} \eta \, dx & m \frac{L^2}{12} + \rho \int_{\frac{L}{2}}^{L} \eta^2 \, dx \end{bmatrix} \quad \gamma_{ref} = \begin{bmatrix} 0 \\ -\rho \sin \theta \int_{\frac{L}{2}}^{L} \eta \, dx - 2 \rho \theta \cos \theta \int_{\frac{L}{2}}^{L} \eta \, dx \\ -\rho \cos \theta \int_{\frac{L}{2}}^{L} \eta \, dx + 2 \rho \theta \sin \theta \int_{\frac{L}{2}}^{L} \eta \, dx \end{bmatrix} \quad B_{ref} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The differential equation of motion of transverse vibration of the beam is expressed as,

$$-\sin \beta \dot{x}_0 + \cos \beta \dot{y}_0 + x \ddot{\theta} + \eta \ddot{\eta} + \frac{\pi}{p} \dot{\eta} = F_{ref} (\theta)$$

(17)

Where, $F_{ref} = \frac{1}{m} [-\sin \beta \cos \theta \dot{x}_0 - \sin \beta \cos \theta \dot{y}]$

It is evident from the eq. (16) and eq. (17) that, rigid and flexible parameters are combined together which shows that beam has rigid as well as flexible motion (vibration).

**MANIPULATOR KINEMATICS AND DYNAMICS**

A well known kinematic relation between the end-effector velocity and joint velocity gives [16],

$$\{ \dot{q} \} = [J] \{ \dot{q}_e \}$$

(18)

Where,

$$J = \begin{bmatrix} J_x & 0 \\ 0 & J_z \end{bmatrix} \quad q = \{ q_x \}$$

Using (5), (18) can be written as,
\[ A_{\text{eff}} = R^T J q \]  
(19)

Where \( R^T \) is the pseudo inverse of \( R \).

General Manipulator dynamic equation can be written in joint space as [16],

\[ M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + G_i(q_i) = T_i + h^T \dot{q}_i \]  
(20)

Where, \( i = 1, 2 \)

For the two manipulators in assembled form,

\[ M_{\text{eff}} \ddot{q} + C_{\text{eff}} \dot{q} + G_{\text{eff}} = \tau + J^T \dot{q} \]  
(21)

Where,

\[ M_{\text{eff}} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad C_{\text{eff}} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad G_{\text{eff}} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad \tau = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \quad J = \begin{bmatrix} J_1 \\ 0 \\ J_2 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \]

COMBINED DYNAMICS

In order to couple the systems such as the two manipulators and a flexible beam, the following procedure is carried out.

Differentiating (19) gives

\[ \ddot{q} = R^T \dot{q} + R^T (\dot{J} \dot{q} + J \ddot{q}) \]  
(22)

Incorporating (19) and (22) into (16) yields the rigid body of the beam in joint space,

\[ M_{\text{eff}} R^T \ddot{q} + M_{\text{eff}} (R^T J + R^T \dot{J}) \dot{q} + C_{\text{eff}} \dot{q} + \eta_{\text{eff}} + G_{\text{eff}} = F_{\text{eff}}(-Q) \]  
(23)

And the corresponding flexible motion is given by,

\[ A_{\text{eff}} R^T \ddot{q} + A_{\text{eff}} (R^T J + R^T \dot{J}) \dot{q} + \eta - \eta^T \ddot{\eta} + \frac{\partial^2 \eta}{\partial q^2} \dot{\eta} \dot{q} = F_{\text{eff}}(Q) \]  
(24)

Where, \( A_{\text{eff}} = [-\sin \theta \sin \theta \cos \theta \theta] \)

Substituting for the force “\( f \)” from (23) into (21) gives the combined dynamics in joint space

\[ M_{\text{eff}} \ddot{q} + C_{\text{eff}} \dot{q} + G_{\text{eff}} + \eta_{\text{eff}} = \tau_{\text{eff}} \]  
(25)

Where,

\[ M_{\text{eff}} = (M_1 + J^T F_{\text{eff}} M_{\text{eff}} R^T) \]

\[ C_{\text{eff}} = C_r + J^T F_{\text{eff}}^r M_{\text{eff}} (R^T J + R^T \dot{J}) \]

\[ G_{\text{eff}} = G_r + J^T F_{\text{eff}}^r G_{\text{eff}} \]

\[ \eta_{\text{eff}} = \eta_{\text{eff}} \]

Since \( F_{\text{eff}} \) is not a square matrix, its inverse \( F_{\text{eff}}^T \) can be calculated by the pseudo inverse.
The complete system of dynamic equations (24) and (25) are coupled with rigid and flexible parameters. Without using any approximate methods, the coupled motions must be controlled. Therefore, singular perturbation approach will be applied in the following section.

**SINGULAR PERTURBATION MODELING IN JOINT SPACE**

In order to develop robust control algorithms, for the system of equations (24) and (25), the following control task is stated.

**Control Task**

For any given desired bounded trajectories \( \mathbf{q}_d \) and \( \mathbf{q}_a \) with some or all of the manipulator and beam parameters unknown, derive a controller for the manipulator control torque \( \tau_{c, \text{ref}} \) such that the manipulator \( \mathbf{q} \) tracks \( \mathbf{q}_d \) while suppressing the vibration of the flexible object, \( \mathbf{v} \) to zero. The above said control task can be achieved by formulating slow and fast subsystem in joint space with the help of singular perturbation technique. In this technique, a two-time scale theory [17] is adapted to identify the slow and fast subsystem dynamics. Then, one can derive the controller for each subsystem, which will be combined to yield the composite control strategy for the original system. However, the challenge in designing the controller is that, stability analysis results should satisfy the Tikhonov’s theorem [18] to guarantee that the composite controller can be applied to the original system. The present analysis considers that the manipulators are rigid and the corresponding inertia matrix \( \mathbf{M}_r \), Coriolis and Centrifugal matrix \( \mathbf{C}_r \), Gravitational vectors \( \mathbf{G}_r \) and Jacobian matrix \( \mathbf{J} \) does not have any flexible parameters. The beam is considered to be flexible, it’s dynamic parameters in the equation (16) such as \( \mathbf{M}_r, \mathbf{C}_r, \mathbf{G}_r, \eta, \mathbf{F}_{\text{ref}} \) and also in \( \mathbf{R} \) contains flexible parameters.

These flexible parameters have to be uncoupled from the above matrices and vectors by using singular perturbation technique. This technique also accounts for the neglected high frequency characteristics when the beam undergoes vibration [19]. Using a perturbation parameter, say \( \varepsilon^2 \), order of the system dynamics can be changed and this small parameter depends upon the system variable. Keeping that in mind, the term \( \text{EI}/\rho \) in (24), which has large magnitude compared to other coefficients, it can be re-defined as,

\[
\frac{\text{EI}}{\rho} \equiv a \cdot K
\]  

Where \( K \) is a dimensionless parameter which has large value for the different materials [19] and its order is equal to \( \text{EI}/\rho \) and also the variable “\( a \)” satisfies the equalities. The beam has rigid motion with respect to the state variables \( \mathbf{X}_{\text{ref}} = \begin{bmatrix} x_0 & y_0 & \theta \end{bmatrix}^T \) and also the transverse vibration \( \eta \) with respect to the state variable occurs in different time scales. Then, one needs to introduce a new variable \( \mathbf{w}(x, t) \) in the same order of the state variable by the following,

\[
\eta(x, t) \equiv \varepsilon^2 \cdot \mathbf{w}(x, t)
\]  

Where \( \varepsilon^2 = 1/K \) is the so-called perturbed parameter.

Substituting (26) and (27) into (24) and (25), the singularly perturbed model of complete system of dynamic equation is derived as,

\[
\mathbf{M}_r \ddot{\mathbf{q}} + \mathbf{C}_r \dot{\mathbf{q}} + \mathbf{F}_{\text{ref}} = \tau_{\text{ref}}
\]  

\[
A_h \dddot{\mathbf{q}} + A_1 \left( \dddot{\mathbf{r}}^T + \dddot{\mathbf{r}}^T \right) \dot{\mathbf{q}} + \varepsilon^2 m \dddot{\mathbf{q}} + \varepsilon^2 m \dddot{\phi}^2 + \omega^2 \mathbf{v} = \mathbf{F}_{\text{ref}}(0)
\]
Where, \( \mathbf{N}_{1_{sd}}, \mathbf{C}_{1_{sd}}, \mathbf{f}_{1_{sd}} \) and \( \mathbf{F}^\dagger \) are obtained by substituting \( w \) instead of \( \eta \).

**SLOW AND FAST DYNAMICS**

The present section considers the typical steps of singular perturbation approach [17] to derive the slow and fast dynamics. Incorporation of perturbed parameter \( \varepsilon^2 \), we will be able to identify the two dynamics in different time scales.

**Slow Dynamics**

When the perturbation parameter \( \varepsilon \) approaches zero (i.e.) \( \varepsilon \to 0 \), in (28) and (29), the equivalent quasi steady state system [18], represents the slow subsystem which is given by,

\[
M_{sd}\ddot{q} + C_{sd}\dot{q} + \mathbf{G}_{sd} = \mathbf{f}_{sd}
\]

(30)

Where,

\[
M_{sd} = \left( \mathbf{M}_2 + J^T \mathbf{F}_{rd}^\dagger \mathbf{M}_{rd} \right)
\]

\[
C_{sd} = \mathbf{G}_2 + J^T \mathbf{F}_{rd}^\dagger \mathbf{M}_{rd} \left( \mathbf{R}_2^{-1} \dot{J} + \mathbf{R}_2^{-1} \right)
\]

\[
\mathbf{G}_{sd} = \mathbf{G}_2 + J^T \mathbf{F}_{rd}^\dagger \mathbf{G}_{rd}
\]

And the transverse vibration of the beam equation becomes,

\[
\left[ A_1 J \mathbf{R}_2^{-1} \dot{q} + A_1 J \left( \mathbf{R}_2^{-1} \dot{J} + \mathbf{R}_2^{-1} \right) \right] \dot{q} + \mathbf{aw}^\dagger \mathbf{q}_{sd} = \mathbf{F}_{sd}(f_{sd})
\]

(31)

Where

\[
F_{rd}^\dagger = \begin{bmatrix}
1 & 0 & \frac{L}{2} \sin \Theta & \frac{L}{2} \cos \Theta \\
0 & 1 & -\frac{L}{2} \sin \Theta & \frac{L}{2} \cos \Theta \\
0 & 0 & 1 & 0 \\
0 & 0 & -\frac{L}{2} \sin \Theta & \frac{L}{2} \cos \Theta
\end{bmatrix}
\]

\[
\mathbf{R}_1^{-1}\]

is the pseudo inverse of \( \mathbf{F}_{rd} \) and \( \mathbf{R}_1 \)

The slow dynamics represented in (30) does not have flexible parameters and hence it corresponds to the rigid body of the complete system. The slow dynamics holds the following properties which will be useful for stability analysis.

**Property 1:** \( \mathbf{N}_{1_{sd}} \) is a symmetric positive definite matrix.

**Property 2:** The matrix \( \mathbf{N}_{1_{sd}} \) and \( \mathbf{C}_{1_{sd}} \) in (30) must satisfies the following,

\[
X^T \left( \mathbf{N}_{1_{sd}} - 2 \mathbf{C}_{1_{sd}} \right) X = 0 \quad \forall X \neq 0
\]

(32)

Where \( X \) is the any arbitrary vector. That is \( \mathbf{N}_{1_{sd}} - 2 \mathbf{C}_{1_{sd}} \) is skew symmetric matrix.
Property 3: There exists a vector \(\mathbf{s}_{\beta}\in\mathbb{R}^{n_{\mathbf{B}}/2}\) which solely depends on manipulator and beam dynamic parameters (lengths, masses and moment of inertias etc.) such that

\[
\mathbf{M}_p \dot{\mathbf{A}} + \mathbf{C}_p \dot{\mathbf{A}} + \mathbf{D}_p = \mathbf{Y}_s \left( \mathbf{A}, \dot{\mathbf{A}}, \mathbf{\tilde{q}} \right) / s_{\mathbf{B}}
\]  

(33)

Where \(\mathbf{Y}_s\in\mathbb{R}^{n_{\mathbf{B}}/2}\) is called regressor of the slow subsystem represented in joint space.

Fast Dynamics

In order to study the vibration of flexible beam, a fast subsystem has to be identified. It is understood from the typical steps of singular perturbation technique that [17], in the fast subsystem slow variables are kept constant and also need to introduce a fast time scale and fast variable.

One can define a fast time scale as

\[
v = t - \frac{\mathbf{z}}{\varepsilon}
\]

(34)

and fast variable as

\[
\mathbf{W}_f = \mathbf{W} - \mathbf{W}_s
\]

(35)

where \(\mathbf{w}_f\) is the fast variable and \(\mathbf{w}_s\) is the slow variable.

Differentiating (34) gives,

\[
\frac{dv}{dt} = \frac{1}{\varepsilon}
\]

(36)

Differentiating (35) yields,

\[
\dot{\mathbf{W}}_f = \dot{\mathbf{W}} - \dot{\mathbf{W}}_s = \dot{\mathbf{W}}_s + \frac{1}{\varepsilon} \dot{\mathbf{z}}
\]

(37)

After differentiating (37),

\[
\dot{\mathbf{W}} = \mathbf{W}_s + \frac{\partial \dot{\mathbf{W}}}{\partial \dot{\mathbf{W}}_s} \frac{d}{dt}\mathbf{z} = \mathbf{W}_s + \frac{1}{\varepsilon^2} \dot{\mathbf{z}}
\]

(38)

Utilizing (35), (37) and (38), (29) becomes,

\[
A_{\mathbf{f}} \mathbf{R}^{f} q + A_{\mathbf{f}} \left( \mathbf{R}^{f} \mathbf{J} + \mathbf{R}^{f} \mathbf{J}^{f} \right) q + \frac{\partial \theta}{\partial \mathbf{z}} + \frac{\partial \mathbf{z}}{\partial \theta} - \varepsilon^2 (\mathbf{w}_f + \mathbf{w}_f^*) \dot{\mathbf{\tilde{q}}} + \left( \mathbf{w}_f + \mathbf{w}_f^* \right) \theta^2 + \frac{\partial \left( \mathbf{w}_f + \mathbf{w}_f^* \right)}{\partial \mathbf{\tilde{q}}} = \mathbf{F}_f(\theta)
\]

(39)

In the boundary layer system, the slow variable \(\mathbf{w}_s\) is constant which implies \(\mathbf{W}_s = \mathbf{0}\) and also \(\varepsilon = 0\). Then, the above equation yields into,

\[
A_{\mathbf{f}} \mathbf{R}^{f} q + A_{\mathbf{f}} \left( \mathbf{R}^{f} \mathbf{J} + \mathbf{R}^{f} \mathbf{J}^{f} \right) q + \frac{\partial \theta}{\partial \mathbf{z}} + \frac{\partial \mathbf{z}}{\partial \theta} = \mathbf{F}_f(\theta)
\]

(40)

Using (31) and also \(\mathbf{F}_f(\mathbf{\tilde{q}}) = \mathbf{F}_f(\theta) - \mathbf{F}_f(\mathbf{\tilde{q}})\) becomes,

\[
\dot{\mathbf{\tilde{q}}} + a \mathbf{w}_f^* = \mathbf{F}_f(\theta)
\]

(41)

Following [21] and [22] an operator \(A\) is defined as,

\[
\mathbf{D}(A) = \{ \mathbf{w}_f, \mathbf{w}_f^* \} \ni \mathbf{w}_f^* (0) = \mathbf{x}_f (0) = \mathbf{x}_f (L) = \mathbf{w}_f^* (L) = \mathbf{0}
\]

(42)
Where, \( D(A) \) denotes the domain of the operator \( A \) and \( H \) denotes the Hilbert space. The operator \( A \) has the eigenvalues \( \lambda_i \) and the corresponding eigen functions \( \phi_i \) satisfying the following conditions:

1) \( 0 < \lambda_1 < \lambda_2 \ldots \ldots \) \( \lim \lambda_i = \infty \)  
2) \( A\phi_i = \lambda_i \phi_i, \ i = 1,2, \ldots \)  
3) The set of the eigen functions forms a complete orthonormal system in the Hilbert space. Utilizing (43), the partial differential equation (41) can be rewritten as an abstract differential equation on \( H \) as,

\[
\begin{align*}
\hat{w}_f (\xi) & = A w_f (\xi) - F_f (\xi) \\
w_f (0) & = w_{f0}, \ w_f (\xi) = w_{f2}
\end{align*}
\] (44)

Equation (44) represents the fast subsystem which will be used for designing fast feedback control.

COMPOSITE CONTROL FOR THE MANIPULATORS – FLEXIBLE OBJECT SYSTEM IN JOINT SPACE

In the previous section, singular perturbation analysis yielded the slow and fast sub-system in joint space. These two subsystems have to be controlled together to achieve the desired trajectory while suppressing the vibrations of the beam. Hence, a composite control law in the following form is considered,

\[
u = u_{s2} (\xi, q, \dot{q}) + u_{cf} (\dot{q}_f, v_f)
\] (45)

Where \( u_{s2} \) is designed based on slow subsystem (30) and the control signal for \( u_{cf} \) is designed for the fast subsystem (44).

Robust Control for Slow Subsystem in Joint Space

In order to handle non linear coupled dynamics and uncertain manipulators and beam parameters, a robust control scheme is considered.

Define the tracking error as,

\[
\varepsilon_{tr} = q - q_d
\] (46)

And the auxiliary trajectory can also be defined as,

\[
\dot{q}_s = \dot{q}_d - \lambda_{s2} \varepsilon_{tr}
\] (47)

Where \( \lambda_{s2} \) is a positive definite matrix whose Eigen values are strictly in the right half of complex plane. The sliding surface can be chosen as,

\[
S_{s2} = q - q_2 = \dot{q}_s + \lambda_{s2} \varepsilon_{tr}
\] (48)

The sliding mode controller can be given as,

\[
u_{s2} = \tau_{s2} = \gamma_{s2} w_{s2} - K_{s2} S_{s2}
\] (49)

Where \( K_{s2} \) is a positive definite gain matrix, \( \gamma_{s2} (\dot{q}_s, \dot{q}_d, q) \) is a regressor matrix and \( w_{s2} = \begin{bmatrix} w_{s1}, \ldots, w_{s2} \end{bmatrix}^T \) are the switching functions which are given by,
\[
\psi_{ij} = -\beta_{ij}\|\alpha_{ij}\|
\]  
(50)

Where \( \beta_{ij} \leq \|\alpha_{ij}\| \) is upper bound of \( \|\alpha_{ij}\| \) which is known though it could be conservatively selected.

**Stability Analysis**

The sliding surface can be written as,
\[
\dot{S}_{ij} = \dot{q}_i - \dot{q}_f
\]  
(51)

Multiplying both sides of (51) by \( M_{ij} \) and using (30), (51) can be rewritten as,
\[
M_{ij}\dot{S}_{ij} = \tau_{ij} - C_{ij}\dot{q}_i - C_{ij}\dot{q}_f - M_{ij}\dot{q}_f
\]  
(52)

Adding and subtracting \( C_{ij}\dot{q}_i \) in (52)
\[
\begin{align*}
M_{ij}\dot{S}_{ij} &= \tau_{ij} - (M_{ij}\dot{q}_i + C_{ij}\dot{q}_i + C_{ij}\dot{q}_i) + C_{ij}\dot{q}_i - C_{ij}\dot{q}_i \\
&= \tau_{ij} - (M_{ij}\dot{q}_i + C_{ij}\dot{q}_i + C_{ij}\dot{q}_i)
\end{align*}
\]  
(53)

By using (48), (53) can be rewritten as,
\[
M_{ij}\dot{S}_{ij} = \tau_{ij} - \tau_{ij}(q_i, q_f) - C_{ij}\dot{S}_{ij}
\]  
(54)

Where,
\[
(M_{ij}\dot{q}_i + C_{ij}\dot{q}_i + C_{ij}\dot{q}_i) = \tau_{ij}(q_i, q_f, q_f)\dot{S}_{ij}
\]

Consider a Lyapunov function candidate as,
\[
V_i(t, S_{ij}) = \frac{1}{2}S_{ij}^T M_{ij} S_{ij}
\]  
(55)

Differentiating (55) with respect to time gives,
\[
\dot{V}_i(t, S_{ij}) = S_{ij}^T M_{ij} \dot{S}_{ij} + \frac{1}{2} S_{ij}^T M_{ij} S_{ij}
\]  
(56)

Substituting (54) into (56) and also using property 2 given in (32), above equation yields,
\[
\dot{V}_i(t, S_{ij}) = S_{ij}^T (\tau - \tau_{ij}(q_i, q_f, q_f)\dot{S}_{ij})
\]  
(57)

Substituting the control law given in (49) and (50) into (57) one can have,
\[
\dot{V}_i(t, S_{ij}) \leq -S_{ij}^T K_{D2} S_{ij} - \beta_{ij}\|\alpha_{ij}\| + \|S_{ij}^T Y_{ij}\|\|\alpha_{ij}\| \leq 0
\]  
(58)

Taking transpose of \( \|S_{ij}^T Y_{ij}\| \) and also \( \beta_{ij} \geq \|\alpha_{ij}\| \) results in,
\[
\dot{V}_i(t, S_{ij}) \leq -S_{ij}^T K_{D2} S_{ij}
\]  
(59)

It is known that \[89\] \( K_{D2} = \rho_{ij} k_1 \) where \( k_1 \) can be considered as a least Eigen value. Using (55), (59) can be rewritten as,
\[
\frac{d^2}{dt^2} \dot{S}_{ij} \leq -2k_1 V_i(t, S_{ij})
\]  
(60)

The solution of the above equation is,
\[ V_0(0, S_0(0)) e^{-\lambda t} \]  

(61)

It is evident from the above equation that the sliding surface will converge exponentially to zero. Thus the sliding surface is related to the tracking error \( e_{tr} \) in (48) which also converges exponentially to zero.

**CONTROL LAW FOR THE FAST DYNAMICS**

The objective of the controller is to suppress the vibration of the flexible object by incorporating following feedback control law. Where \( F^t_{tr} \) can be found using pseudo inverse. The operator \( \Pi \) is neither self adjoint nor positive definite and is also shown in [20] and [22] that, it is A-symmetric and A-positive semi definite. This operator was formulated in [22] as \( \Pi = kQ^t A \) where \( Q \) is a bounded and positive definite operator. Also, the velocity signal \( \Psi_1(\nu) \) can be measured using velocity sensor. It is to be noted that, there are some established results are available using the velocity feedback for example in [23] and [24]. However, they have considered the assumption of modes where in this control algorithm no form of approximation is used.

Substituting (62) into (44) gives a closed loop system defined by,

\[ \dot{\Psi_1}(\nu) + \Pi \Psi_1(\nu) + A w_2(\nu) = 0 \]

(63)

Utilizing the operators \( \Pi \) and \( Q \), (63) can be rewritten as,

\[ \dot{\Psi_1}(\nu) + kQ^t A \Psi_1(\nu) + A w_2(\nu) = 0 \]

\[ \nu_2(0) = \nu_2, \quad \nu_1(0) = w_2 \]

(64)

Where \( k \) is the positive gain and the term \( kQ^t A \Psi_1(\nu) \) is the special damping term [22]. The two operators \( Q \) and \( A \) are related by \( Q = A^t \) and \( \beta \) varies between \( \left( -\frac{1}{2}, 0 \right) \). It is shown analytically by Haung [25] that when \( \beta = -\frac{1}{2} \), the damping term \( Q^t A \Psi_1(\nu) \) becomes \( \frac{1}{2} \Psi_1(\nu) \), this corresponds to structural damping which can be also seen in [26]. If \( \beta = 0 \), then damping term exhibits strong damping or over damping characteristics [25].

**Stability Analysis**

**Exponential Stability of Fast Subsystem:** Tikhonov’s theorem requires that the fast subsystem also must be exponentially stable. The energy multiplier method used in [22] is followed to prove the exponential stability under the following theorem [27], (theorem 4.1) which guarantees exponential stability.

**Theorem:** Let A be the infinitesimal generator of a \( C_0 \) semi group \( T(t) \). If for some \( p, \, 1 \leq p \leq \infty \) then there are constants \( M \geq 1 \) and \( \mu > 0 \) such that \( \| T(t) \| \leq Me^{-\mu t} \).

\[ \int_0^\infty \| T(t) \|^p dt < \infty \]

(65)

Note: It is also shown in [22] that the property of \( L^p \) stability and exponential stability for a strong continuous semi group must satisfy the following,

\[ \int_0^\infty \| E(t) \|^2 dt < \infty \]

(66)
Proof: Let the energy function for (64) be of the form,

\[ E(v) = \frac{1}{2} \| A \dot{\phi}_1(v) \|^2 + \frac{1}{2} \| A^2 \dot{\phi}_2(v) \|^2 \]  

(67)

Where \( E(v) \) is weakly monotonically decreasing function with respect to fast time scale \( v \). The fast time scale derivative of \( E(v) \) from the above equation will be,

\[ \dot{E}(v) = \frac{1}{2} \frac{1}{k(QA \dot{\phi}_1(v))} \leq 0 \]  

(68)

Let us choose \( 0 < \varepsilon < 1 \) and the Lyapunov function candidate is given by,

\[ V_2(v) = 2(1 - \varepsilon)v \dot{E}(v) + (\dot{\phi}_1(v)QA \dot{\phi}_1(v)) \]  

(69)

We have the following relation,

\[ (\dot{\phi}_1(v)QA \dot{\phi}_1(v)) \leq \frac{1}{2} \left( \| A \dot{\phi}_1(v) \|^2 + \| A^2 \dot{\phi}_2(v) \|^2 \right) \]

There exists a constant \( C_1 \) such that,

\[ (2(1 - \varepsilon)v - C_2)E(v) \leq V_2(v) \leq (2(1 - \varepsilon)v + C_2)E(v) \]  

(70)

For \( v > v_1 \) the Lyapunov function is positive and \( v_1 \) is from,

\[ 2(1 - \varepsilon)v - C_2 = 0 \]  

(71)

The derivative of \( V_2(v) \) with respect to fast time scale is given by,

\[ V_2'(v) = \left( (2 - \varepsilon) \| A^2 \dot{\phi}_1(v) \|^2 - \varepsilon \| A \dot{\phi}_1(v) \|^2 - \frac{k}{2} \| QA \dot{\phi}_1(v) \| A \dot{\phi}_1(v) \|^2 - k(1 - \varepsilon) \| QA \dot{\phi}_1(v) \|^2 \right) \]  

(72)

For any arbitrary constant, say, \( C_2 > 0 \) we have,

\[ -(QA \dot{\phi}_1(v)QA \dot{\phi}_1(v)) \leq \frac{1}{2} \lambda_{\text{max}}(Q) \left( C_2 \| QA \dot{\phi}_1(v) \|^2 + \frac{k}{2} \| QA \dot{\phi}_1(v) \| A \dot{\phi}_1(v) \| A \dot{\phi}_1(v) \|^2 \right) \]  

(73)

Using (73), (72) can be rewritten as,

\[ V_2'(v) = \left( (2 - \varepsilon) \| A^2 \dot{\phi}_1(v) \|^2 - \frac{k}{2} \lambda_{\text{max}}(Q) - 2k(1 - \varepsilon) \lambda_{\text{min}}(Q) \| A \dot{\phi}_1(v) \|^2 \right) \]  

(74)

Where \( \lambda_{\text{max}}(Q) = \max_{\tilde{w} \in \tilde{H}(Q \tilde{W}, \tilde{W})} \) and \( \lambda_{\text{min}}(Q) = \min_{\tilde{w} \in \tilde{H}(Q \tilde{W}, \tilde{W})} \).

If \( C_2 \) can be chosen as large then \( \left( \varepsilon - \frac{k}{2C_2} \lambda_{\text{max}}(Q) \right) > 0 \),

\[ V_2(v) \leq 0 \quad \forall v > v_2 \]  

(75)

Where \( v_2 \) can be,

\[ (2 - \varepsilon) \| A^2 \dot{\phi}_1(v) \|^2 - \frac{k}{2} \lambda_{\text{max}}(Q) - 2k(1 - \varepsilon) \lambda_{\text{min}}(Q) = 0 \]
The above result in (75) shows that derivative of Lyapunov function has decreasing nature for \( v < v_2 \) and it is also evident from (68) that the energy will also be dissipating for \( v > 0 \). Using these facts, for \( v > T = \text{max} \{v_1, v_2\} \) and also from (70), \( E(v) \) can be estimated as,

\[
E(v) \leq \frac{V_0}{2n - 2n - 2n - 2n} \frac{(v_1 - 2T + d)E(v)}{2(2n - 2n - 2n - 2n)}
\]  

(76)

Then, (76) can be rewritten as,

\[
\int_0^\infty E(v) \, dv \leq \int_0^\infty (v_1 - 2T + d)^2 E(v) \, dv
\]

(77)

Which confirms the exponential stability and hence it is proved.

**SIMULATION RESULTS**

The objective of this composite controller is to move the object from the initial position of center of mass and orientation (0.51m; 0.36m; 90°) to the final position and orientation (0.55 m; 0.36 m; 90°) using two planar manipulators each with three links, while suppressing the vibrations. The object motion corresponds to move each revolute joint of first manipulator from (0; -45°; -45°) to (-10.35°; -21.5°; -58.2°) and correspondingly the second manipulator from the initial angular position (0°; 45°; 45°) to final position (10.35°; 21.5°; 58.2°). The beam is of 0.1m length and mass of 1 Kg.

<table>
<thead>
<tr>
<th>Link</th>
<th>Length (M)</th>
<th>Mass (Kg)</th>
<th>Moment of Inertia (Kg.m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>1.0</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>1.0</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>0.4</td>
<td>0.15</td>
</tr>
</tbody>
</table>

It is shown in Fig. 2 that, beam approaches towards its final position along X direction within 0.5 secs. The translation along Y direction and orientation of the beam are also maintained towards their desired values which are shown in Figs. 3 and 4. Due to the highly nonlinear manipulator parameters, a small deviation to the final value occurs initially in Figs. 3 and 4 and after 0.2 secs the beam center has reached its final pose. The two manipulators are also reached the initial position to the final position as shown in the Figs. 5-10. The flexible motion of the beam is plotted for damping ratios. For the damping ratio 0.1, the different mode shapes are shown in Fig. 11.
Figure 4: Orientation of the Beam

Figure 5: Manipulator 1 Joint 1 Motion

Figure 6: Manipulator 1 Joint 2 Motion

Figure 7: Manipulator 1 Joint 3 Motion

Figure 8: Manipulator 2 Joint 1 Motion

Figure 9: Manipulator 2 Joint 1 Motion
Figure 10: Manipulator 2 Joint 3 Motion
Figure 11: Deflection Control of the Beam

The structural damping behavior is shown in Figure 12. It is observed from the simulation results that the beam is achieved desired rigid body motion and the vibration is also suppressed.

Figure 12: Structural Damping Behavior of the Beam

CONCLUSIONS

In this article, the problem of two planar rigid manipulators used to move a flexible object in the desired position and orientation while suppressing the vibration of the flexible object is considered. The kinematic and dynamic relations of the beam are derived without either approximating or discretizing the beam. The complete system is then obtained by combining both the dynamics of the beam and the manipulators. To control this highly nonlinear system, singular perturbation method is adopted, where the slow and fast subsystems are separately obtained. Based on the Tikhonov’s theorem, a regressor based sliding mode control algorithm is designed for the slow subsystem and a simple feedback control law is selected for fast subsystem to form a composite controller. The simulation results show that the proposed composite controller is an efficient choice in achieving the regulation control performance while suppressing the vibration of the flexible object.
REFERENCES


