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STATISTICAL APPROXIMATION FOR GENUINE MIXED INTEGRAL TYPE OPERATORS

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ABSTRACT

This paper deals with the study the q-analogue of integral type operators namely Szasz-Mirakyan-Baskakov-Stancu operators. For finding the moments, we apply q-derivative and q-Beta functions of these operators. We also examine some estimations e.g. weighted approximation, rate of convergence and point-wise convergence.

AMS Subject Classification: 41A25, 41A30

KEY WORDS: Baskakov Operators, q-Beta Function, q-Analogue & Weighted Approximation

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1. INTRODUCTION

Foundation

In approximation theory, familyof linear positive operators with the combination of the well known Szasz-Mirakyan operators and Baskakov operators was established by Gupta and Srivastava [10]. These operators are defined as follows

$$S_n(f,x) = (n-1)\sum_{u=0}^{\infty} l_{n,u}(x)\int_0^{\infty} p_{n,u}(t)f(t)dt, x \in [0,\infty) \tag{1.1}$$

where, $l_{n,u}(x) = \frac{e^{-nx}(nx)^u}{u!}, p_{n,u}(t) = \binom{n+u-1}{u}.\frac{t^u}{(1+t)^{n+u}}$.

Later on, other approximation properties on such integral type operators were studied by several other researchers such as [7], [9], [12], [18] and [19] etc. In [1], authors have given the rate of convergence for functions including derivatives of bounded variation. Since last few years, application of q-calculus has been a wide area of research for positive linear operators. Manyq-types of generalizations ofintegral type operators and their approximation behaviors were intensively described.

q-analogue of various integral operators such as Szasz -Mirakyan operators, Szasz -Mirakyan Beta operators and other operators were proposed by [2], [3], [8], [15], [17], [16].

q-analogue of above operators were proposed and established by [11] as follows

$$S_n^q(f(t),x) = [n-1]_q \sum_{u=0}^{\infty} l_{n,u}^q(x) q^u \int_0^{\infty/A} p_{n,u}^q(t) f(t) d_q t, \tag{1.2}$$

where $l_{n,u}^q(x) = \frac{([n]_q x)^u}{[u]_q!} q^{\frac{u(u-1)}{2}} \frac{1}{E_q([n]_q x)}$ and $p_{n,u}^q(t) = \begin{bmatrix} n+u-1 \\ u \end{bmatrix}_q q^{\frac{u(u-1)}{2}} \frac{t^u}{(1+t)_q^{n+u}}$.

For $\alpha \in [0,\beta]$ and $n \in N, 0 < q < 1$, the Stancu- generalization of above operators (1.2) are given below

$$S_{n,\alpha,\beta}^{q}(f(t),x) = [n-1]_q \sum_{u=0}^{\infty} l_{n,u}^{q}(x) q^u \int_0^{\infty/A} p_{n,u}^{q}(t) f\left(\frac{[n]_q t+\alpha}{[n]_q t+\beta}\right) d_q t, \tag{1.3}$$

Where $l_{n,u}^{q}(x)$ and $p_{n,u}^{q}(t)$ are defined above.

Remark

Operators defined in equations (1.1) and (1.2) are linear, when $q = 1$, $\alpha = 0$ and $\beta = 0$, then operators (1.3) become the operators discussed in [7].

Here, we specify some notations of q-calculus, these notations can also be studied in [4], [13].

We have, $[n]_q = \frac{1-q^n}{1-q}$, for every $n \in N$.

The q-derivative $D_q f$ of a function f is given by

$$\left(D_q f\right)(x) = \frac{f(x)-f(qx)}{(1-q)x}, x \neq 0 \tag{1.4}$$

The q-proper integral are defined as (see [13])

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, A > 0$$

and $\int_0^{\infty} f(x) d_q x = (1-q) a \sum_{n=-\infty}^{\infty} f(aq^n)\, q^n, a > 0$ **(1.5)**

We have given some auxiliary propositions named as lemmas, which are very helpfulfor the proof of our main results. We obtain approximation behavior of the operators (1.3) in the form of second order modulus of smoothness and classical modulus of continuity. We also present uniform convergence theorems via weighted approximation for the functions belonging to weighted spaces. Lastly, we prove the point-wise estimates for the functions satisfying the Lipschitz conditions.

2. AUXILIARY RESULTS

In this section we prove some auxiliary propositions (lemmas)

Lemma 1. [11] Moments for these operators are given as

- $S_n^q(1,x) = 1$
- $S_n^q(t,x) = \frac{[n]_q}{q^2[n-2]_q} + \frac{1}{q[n-2]_q}, n > 2$
- $S_n^q(t^2,x) = \frac{[n]_q^2}{q^6[n-2]_q[n-3]_q} x^2 + \frac{[n]_q[1+q]^2}{q^5[n-2]_q[n-3]_q} x + \frac{[2]_q}{q^3[n-2]_q[n-3]_q},$

$$= \frac{[n]_q^2 x^2 + q[n]_q[1+q]^2 x + q^3[2]_q}{q^6[n-2]_q[n-3]_q}, n > 3.$$

Lemma 2. The central moments are given as

$\mu_{n,m}^{\alpha,\beta}(x) = S_{n,\alpha,\beta}^{q}(t^m,x) = [n-1]_q \sum_{u=0}^{\infty} l_{n,u}^{q}(x) q^u \int_0^{\infty/A} p_{n,u}^{q}(t) f\left(\frac{[n]_q t+\alpha}{[n]_q t+\beta}\right)^m d_q t$, then

- $\mu_{n,0}^{\alpha,\beta}(x) = S_{n,\alpha,\beta}^{q}(1,x) = 1$
- $\mu_{n,1}^{\alpha,\beta}(x) = S_{n,\alpha,\beta}^{q}(t,x) = \frac{[n]_q^2}{q^2[n-2]_q([n]_q+\beta)}x + \frac{[n]_q}{q[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)}, n > 2$
- $\mu_{n,2}^{\alpha,\beta}(x) = S_{n,\alpha,\beta}^{q}(t^2,x) = \left(\frac{[n]_q}{([n]_q+\beta)}\right)^2 \left[\frac{[n]_q^2x^2+q[n]_qx[1+q]^2+[2]_qq^3}{q^6[n-2]_q[n-3]_q}\right]$

$$+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}\left[\frac{[n]_qx+q}{q^2[n-2]_q}\right] + \left(\frac{[n]_q}{([n]_q+\beta)}\right)^2, n > 3.$$

Proof: Following [11], we have$\mu_{n,0}^{\alpha,\beta}(x) = 1$,

$$\mu_{n,1}^{\alpha,\beta}(x) = [n-1]_q \sum_{u=0}^{\infty} l_{n,u}^{q}(x) q^u \int_0^{\infty/A} p_{n,u}^{q}(t) f\left(\frac{[n]_qt+\alpha}{[n]_qt+\beta}\right) d_qt$$

$$= \frac{[n]_q}{([n]_q+\beta)} S_n^q(t,x) + \frac{\alpha}{([n]_q+\beta)} S_n^q(1,x) = \frac{[n]_q^2}{q^2[n-2]_q([n]_q+\beta)}x + \frac{[n]_q}{q[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)},$$

also, $\mu_{n,2}^{\alpha,\beta}(x) = [n-1]_q \sum_{u=0}^{\infty} l_{n,u}^{q}(x) q^u \int_0^{\infty/A} p_{n,u}^{q}(t) f\left(\frac{[n]_qt+\alpha}{[n]_qt+\beta}\right)^2 d_qt$

$$= \left(\frac{[n]_q}{([n]_q+\beta)}\right)^2 S_n^q(t^2,x) + \frac{2\alpha[n]_q}{([n]_q+\beta)^2} S_n^q(t,x) + \left(\frac{\alpha}{([n]_q+\beta)}\right)^2 S_n^q(1,x)$$

$$= \left(\frac{[n]_q}{([n]_q+\beta)}\right)^2 \left[\frac{[n]_q^2x^2+q[n]_qx[2]_q^2+[2]_qq^3}{q^6[n-2]_q[n-3]_q}\right] + \frac{2\alpha[n]_q}{([n]_q+\beta)^2}\left[\frac{[n]_qx+q}{q^2[n-2]_q}\right] + \left(\frac{\alpha}{([n]_q+\beta)}\right)^2.$$

Corollary 1. Using Lemma 2, we set

$$S_{n,\alpha,\beta}^{q}((t-x)^2,x) = \left(\frac{[n]_q^4}{q^6[n-2]_q[n-3]_q([n]_q+\beta)^2} - \frac{2[n]_q^2}{q^2[n-2]_q[n-3]_q([n]_q+\beta)} + 1\right)x^2$$

$$+\left(\frac{[n]_q^3[2]_q^2}{q^5[n-2]_q[n-3]_q([n]_q+\beta)^2} + \frac{2[n]_q^2\alpha}{q^2[n-2]_q[n-3]_q([n]_q+\beta)^2} - \frac{2[n]_q}{q[n-2]_q([n]_q+\beta)} - \frac{2\alpha}{([n]_q+\beta)}\right)x$$

$$+\frac{[n]_q^2[2]_q}{q^3[n-2]_q[n-3]_q([n]_q+\beta)^2} + \frac{2\alpha[n]_q}{q[n-2]_q([n]_q+\beta)^2} + \frac{\alpha^2}{([n]_q+\beta)^2} = \delta_n(q,x)$$

and$\gamma_n(q,x) = \frac{([2]_q[n]_q-q^2\beta[n-2]_q)x+q[n]_q+\alpha q^2[n-2]_q}{q^2[n-2]_q([n]_q+\beta)}$.

3. MAIN RESULTS

Definition: Let $C_B[0,\infty)$denotes the space of all functions f which are continuous and bounded on the interval$[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

We define the Peetre's K-functional as, $K_2(f,\delta) = \inf_{j\in C_B^2[0,\infty)} \{\|f-j\| + \delta\|j''\|\}$,

where, $C_B^2[0,\infty) = \{j \in C_B[0,\infty): j', j'' \in C_B[0,\infty)\}$.By [5], ∃ an absolute positive constant "C" such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta}), \quad \textbf{(3.1)}$$

where $\delta > 0$, we write the second order modulus of smoothness as below

$$\omega_2(f,\delta) = \sup_{0\le h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

In this continuation the usual modulus of continuity of $f \in C_B[0,\infty)$ is given as

$$\omega(f,\delta) = \sup_{0\le h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|. \qquad \textbf{(3.2)}$$

Theorem 1. Let $f \in C_B[0,\infty)$, then for all $j \in C_B^2[0,\infty)$, we have

$$\left|\bar{S}_{n,\alpha,\beta}^{q}(j,x) - j(x)\right| \le (\delta_n(q,x) + \gamma_n^2(q,x))\|j''\|, \qquad \textbf{(3.3)}$$

where,$\bar{S}_{n,\alpha,\beta}^{q}(f,x) = S_{n,,\alpha,\beta}^{q} + f(x) - f\left(\frac{[n]_q^2 x}{q^2[n-2]_q([n]_q+\beta)}x + \frac{[n]_q}{q[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)}\right)$. **(3.4)**

Proof: From (3.4) and Lemma 2, we have

$$\overline{S}_{n,\alpha,\beta}^{q}(f,x)(t-x,x) = 0 \qquad \textbf{(3.5)}$$

Suppose $x \in [0,\infty)$ and $j \in C_B^2[0,\infty)$. Using Taylor's expansion

$$j(t) - j(x) = (t-x)j'(x) + \int_x^t (t-y)j''(y)dy.$$

Applying the operators $\bar{S}_{n,\alpha,\beta}^{q}$ to both the sides of above equality and taken in to account (3.5), we obtain

$$\bar{S}_{n,\alpha,\beta}^{q}(j,x) - j(x) = \bar{S}_{n,\alpha,\beta}^{q}\big((t-x)j'(x),x\big) + \bar{S}_{n,\alpha,\beta}^{q}\left(\int_x^t (t-y)j''(y)dy, x\right)$$

$$= j'(x)\bar{S}_{n,\alpha,\beta}^{q}\big((t-x)j'(x),x\big) + S_{n,\alpha,\beta}^{q}\left(\int_x^t (t-y)j''(y)dy, x\right)$$

$$-\int_x^{\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)}x+\frac{\alpha}{([n]_q+\beta)}}\left(\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)} - y\right)j''(y)dy$$

$$= S_{n,\alpha,\beta}^{q}\left(\int_x^t (t-y)j''(y)dy, x\right) - \int_x^{\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)}x+\frac{\alpha}{([n]_q+\beta)}}\left(\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)} - y\right)j''(y)dy.$$

We get, $\left|\int_x^t (t-y)j''(y)dy\right| \le \left|\int_x^t (t-y)|j''(y)|dy\right| \le \|j''\|\left|\int_x^t (t-y)dy\right| \le (t-x)^2|j''|$, and

$$\left|\int_x^{\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)}x+\frac{\alpha}{([n]_q+\beta)}}\left(\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)} - y\right)j''(y)dy\right|$$

$$\le \left(\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)} - y\right)^2\|j''\| = \gamma_n^2(q,x)\|j''\|.$$

We follow that, $\left|\bar{S}_{n,\alpha,\beta}^{q}(j,x) - j(x)\right| = |\bar{S}_{n,\alpha,\beta}^{q}\left(\int_x^t (t-y)j''(y)dy, x\right)$

$$-\int_x^{\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)}x+\frac{\alpha}{([n]_q+\beta)}}\left(\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)} - y\right)j''(y)dy|$$

$$\le S_{n,,\alpha,\beta}^{q}((t-x)^2\|j''\|,x) + \left(\frac{[n]_q^2x+q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)} - x\right)^2\|j''\| = \delta_n(q,x) + \gamma_n^2(q,x)\|j''\|.$$

Theorem 2. Suppose$f \in C_B[0,\infty)$, then for all$\in [0,\infty), \exists$ a constant $K > 0$, such that

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \le K\omega_2\big(f, \sqrt{\delta_n(q,x) + \gamma_n^2(q,x)}\big)\omega\big(f,\gamma_n(q,x)\big)$$

Proof. From (3.4), for $j \in C_B^2[0,\infty)$, we can get

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \le \left|\bar{S}_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| + \left|f(x) - f\left(\frac{[n]_q^2 x + q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)}\right)\right|$$

$$\le \left|\bar{S}_{n,\alpha,\beta}^{q}(f,-j,x) - (f-j)(x)\right| + \left|f(x) - f\left(\frac{[n]_q^2 x + q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)}\right)\right| + \left|\bar{S}_{n,\alpha,\beta}^{q}(j,x) - j(x)\right|$$

$$\le \left|\bar{S}_{n,\alpha,\beta}^{q}(f,-j,x)\right| + |(f-j)(x)| + \left|f(x) - f\left(\frac{[n]_q^2 x + q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)}\right)\right| + \left|\bar{S}_{n,\alpha,\beta}^{q}(j,x) - j(x)\right|.$$

Taking boundedness of the operators $\bar{S}_{n,\alpha,\beta}^{q}$ and using (3.3), we get

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \le 4\|f-j\| + \left|f(x) - f\left(\frac{[n]_q^2 x + q[n]_q}{q^2[n-2]_q([n]_q+\beta)} + \frac{\alpha}{([n]_q+\beta)}\right)\right|$$

$$+\big(\delta_n(q,x) + \gamma_n^2(q,x)\big)\|j''\| \le 4\|f-j\| + \big(\delta_n(q,x) + \gamma_n^2(q,x)\big)\|j''\| + \omega\big(f,\gamma_n(q,x)\big).$$

Now over all$j \in C_B^2[0,\infty)$, considering infimum on right hand side and applying equation (3.1), we follow

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \le 4K_2\big(f,\delta_n(q,x) + \gamma_n^2(q,x)\big) + \omega\big(f,\gamma_n(q,x)\big)$$

$$\le 4M\omega_2\big(f,\sqrt{\delta_n(q,x) + \gamma_n^2(q,x)}\big) + \omega\big(f,\gamma_n(q,x)\big) \le K\omega_2\big(f,\sqrt{\delta_n(q,x) + \gamma_n^2(q,x)}\big) + \omega\big(f,\gamma_n(q,x)\big),$$

where *K=4M>0.*

Theorem 3. Let f be a function which is bounded and integrable on the interval $[0,\infty)$. For $q = q_n \in (0,1)$,suppose second order derivative offunction f exists at a fixed point $x \in [0,\infty)$, such that $q_n \to 1$ as $n \to \infty$, than

$$\lim_{n\to\infty}[n]_{q_n}\left[S_{n,\alpha,\beta}^{q_n}(f,x) - f(x)\right] = [(2-\beta)x + 1 - \alpha]f'(x) + \left(\frac{x^2}{2} + x\right)f''(x).$$

Proof: For the proof of above theorem, we use Taylor's expansion

$$f(t) - f(x) = (t-x)f'(x) + (t-x)^2\left(\frac{1}{2}f''(x) + \varepsilon(t-x)\right),$$

Where ε is bounded and$\lim_{t\to 0}\varepsilon(t) = 0$. Now applying the operators $S_{n,\alpha,\beta}^{q}(f,x)$ to the above relation, we have

$$S_{n,\alpha,\beta}^{q}(f,x) - f(x) = f'(x)S_{n,\alpha,\beta}^{q_n}\big((t-x),x\big) + \frac{1}{2}f''(x)S_{n,\alpha,\beta}^{q_n}((t-x)^2,x) + S_{n,\alpha,\beta}^{q_n}(\varepsilon(t-x)(t-x)^2,x)$$

$$= f'(x)\gamma_n(q_n,x) + \frac{1}{2}f''(x)\delta_n(q_n,x) + S_{n,\alpha,\beta}^{q_n}(\varepsilon(t-x)(t-x)^2,x),$$

where $\gamma_n(q_n,x)$ and $\delta_n(q_n,x)$ can be seen in equation (3.2).

Now using inequality of Cauchy-Schwarz, we have

$$[n]_q S_{n,\alpha,\beta}^{q_n}(\varepsilon(t-x)(t-x)^2,x) \le \left(S_{n,\alpha,\beta}^{q_n}(\varepsilon^2(t-x))\right)^{1/2}\left([n]_{q_n}^2 S_{n,\alpha,\beta}^{q_n}((t-x)^4,x)\right)^{1/2}$$

Using Lemma 2, we have$\lim_{n\to\infty}[n]_{q_n}^2 S_{n,\alpha,\beta}^{q_n}((t-x)^4,x) = 0,$

since $\lim_{n\to\infty}\gamma_n(q_n,x) = (2-\beta)x + 1 - \alpha$ and $\lim_{n\to\infty}\alpha_n(q_n,x) = x^2 + 2x$.

This is the required result.

Definition 2: Let $H_{x^2}[0,\infty)$represents the set of all available functions *f*which are defined on the interval $[0,\infty)$, satisfying the condition $|f(x)| \le M_f(1+x^2)$, where M_f is a constant depending only on f and $C_{x^2}[0,\infty)$ is the subspace of all functions which are continuous and belongs to $H_{x^2}[0,\infty)$. Let $C^*_{x^2}[0,\infty)$ is the subspace of all functions$f$ belonging to$C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite.We define norm on $C^*_{x^2}[0,\infty)$ as below

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}.$$

Theorem 4. Let $q = q_n$ satisfies $q_n \in (0,1)$ and let $q_n \to 1\ as\ n\to\infty$. Then for each $f \in C_{x^2}[0,\infty)$, we have

$$\lim_{n\to\infty}\left\|S^{q_n}_{n,\alpha,\beta}(f) - f\right\|_{x^2} = 0.$$

Proof: Following the weighted Korokvin theorem in [6], we see it will be enough to verify the following$\lim_{n\to\infty}\left\|S^{q_n}_{n,\alpha,\beta}(t^k,x) - x^k\right\|_{x^2} = 0$, where k is a positve integer and$k \in [0,2]$. **(3.6)**

Since $S^{q_n}_{n,\alpha,\beta}(1,x) = 1$, so theorem holds for $k = 0$, to get the proof of the theorem, it is suffices to show that

$$S^{q_n}_{n,\alpha,\beta}(t^k,x) = x^k, k = 1, 2.$$

Now $\left\|S^{q_n}_{n,\alpha,\beta}(t,x) - x\right\|_{x^2} \le \sup_{x\in[0,\infty)}\frac{([n]_q^2 - q^2[n-2]_q([n]_q+\beta)x + q[n]_q + \alpha q^2[n-2]_q}{q^2[n-2]_q([n]_q+\beta)}\frac{1}{1+x^2}$

$$\le \frac{([2]_q[n]_q[-q^2[n-2]_q\beta) + q[n]_q + \alpha q^2[n-2]_q}{q^2[n-2]_q([n]_q+\beta)}\sup_{x\in[0,\infty)}\frac{x+1}{1+x^2},$$

which implies $\lim_{n\to\infty}\left\|S^{q_n}_{n,\alpha,\beta}(t,x) - x\right\|_{x^2} = 0$.

Also $\left\|S^{q_n}_{n,\alpha,\beta}(t^2,x) - x^2\right\|_{x^2} \le \sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\left(\frac{[n]_q}{([n]_q+\beta)}\right)^2\left(\frac{[n]_q^2}{q^6[n-2]_q[n-3]_q} - \left(\frac{([n]_q+\beta)}{[n]_q}\right)^2\right)$

$$+\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left[\frac{[n]_q x[2]_q^2 + q^2[2]_q}{q^5[n-2]_q[n-3]_q}\right] + \sup_{x\in[0,\infty)}\frac{1}{1+x^2}\frac{2[n]_q\alpha}{([n]_q+\beta)}\left[\frac{[n]_q x + q}{q^2[n-2]_q}\right] + \sup_{x\in[0,\infty)}\frac{1}{1+x^2}\left(\frac{\alpha}{([n]_q+\beta)}\right)^2$$

$$= \left(\frac{\alpha}{([n]_q+\beta)}\right)^2\left(\frac{[n]_q^2}{q^6[n-2]_q[n-3]_q} - \left(\frac{([n]_q+\beta)}{[n]_q}\right)^2\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2},$$

$$+\left[\frac{[n]_q(1+q)^2}{q^5[n-2]_q[n-3]_q} + \frac{2[n]_q\alpha}{([n]_q+\beta)^2}\frac{[n]_q}{q^2[n-2]_q}\right]\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$+\left[\left(\frac{\alpha}{([n]_q+\beta)}\right)^2 + \frac{[2]_q}{q^3[n-2]_q[n-3]_q} + 2[n]_q\frac{\alpha}{([n]_q+\beta)^2}.\frac{1}{q[n-2]_q}\right]\sup_{x\in[0,\infty)}\frac{1}{1+x^2},$$

which implies that $\lim_{n\to\infty}\left\|S^{q_n}_{n,\alpha,\beta}(t^2,x) - x^2\right\|_{x^2} = 0$.

Hence the proof.

Theorem 5. Suppose $\alpha \in (0,1]$ and let Sdenotes any bounded subset of$[0,\infty)$. If $f \in C_B(0,\infty) \cap LipM(\alpha)$, then we know,$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \leq K\left\{\delta_n^{\frac{\alpha}{2}}(q,x) + 2(d(x,S))^{\alpha}\right\}$,

where K is an arbitrary constant which depends on α and $d(x, S)$ being the distance between x and subset S, which can be defined as $d(x,S) = inf\{(t-x): t \in S and x \in [0,\infty)\}$ and $\delta_n(q,x)$ are defined in Corollary 1.

Proof: Following the properties related to infimum, there exists at least one point y in the closure of S,

$y \in S$: $d(x,S) = |y - x|$.

Taking in view the triangular inequality, we get$|f(t) - f(x)| \leq \left|f(t) - f(y) + |f(y) - f(x)|\right|$

Hence, $\left|S_{n,\alpha,\beta}^{q}(t,x) - f(x)\right| \leq S_{n,\alpha,\beta}^{q}(|f(t) - f(x)|, x)$

$\leq S_{n,\alpha,\beta}^{q}(t,x)(|f(t) - f(y)|, x) + S_{n,\alpha,\beta}^{q}(|f(y) - f(x)|, x)$

$\leq K\left\{S_{n,\alpha,\beta}^{q}(|t-y|^{\alpha}, x) + |y-x|^{\alpha}\right\} \leq K\left\{S_{n,\alpha,\beta}^{q}(|t-x|^{\alpha} + |x-y|^{\alpha}, x) + |y-x|^{\alpha}\right\}$

$\leq K\left\{S_{n,\alpha,\beta}^{q}(|t-x|^{\alpha}, x) + 2|y-x|^{\alpha}\right\}$.

We opt$l_1 = \frac{2}{\alpha}, l_2 = \frac{2}{2-\alpha}$ *and we get* $\frac{1}{l_1} + \frac{1}{l_2} = 1$, and then using Holder's inequality, we have

$$\left|S_{n,\alpha,\beta}^{q}(t,x) - f(x)\right| \leq K\left\{\left[S_{n,\alpha,\beta}^{q}(|t-x|^{\alpha l_1}, x)\right]^{1/l_1} \times \left[S_{n,\alpha,\beta}^{q}(1^{l_2}, x)\right]^{1/l_2} + 2|y-x|^{\alpha}\right\}$$

$$= K\left\{\left[S_{n,\alpha,\beta}^{q}(|t-x|^{2}, x)\right]^{\alpha/2} + 2|y-x|^{2}\right\} = K\left\{\delta_n^{\frac{\alpha}{2}}(q,x) + 2(d(x,S))^{\alpha}\right\}.$$

Hence, we get the proof of the theorem.

Now, to find local direct estimates of the operators defined in (1.3), following [14] and using Lipcshitz type maximal function of order α.

$$\bar{\omega}_{\alpha}(f,x) = \sup_{t \neq x, t \in [0,\infty)} \frac{|f(t) - f(x)|}{|t-x|^{\alpha}}, x \in (0,\infty] \text{ and } \alpha \in (0,1]. \quad \textbf{(3.7)}$$

Theorem 6. Let $f \in C_B[0,\infty)$, and $0 \leq \alpha \leq 1$, then for every $x \in (0,\infty]$, we have

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \leq \bar{\omega}_{\alpha}(f,x)\delta_n^{\frac{\alpha}{2}}(x).$$

Proof: From the equation (3.7), we have

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \leq \bar{\omega}_{\alpha}(f,x) S_{n,\alpha,\beta}^{q}(t,x)(|t-x|^{\alpha}, x).$$

Applying the Holder inequality, with$l_1 = \frac{2}{\alpha}, l_2 = \frac{2}{2-\alpha}$, we get

$$\left|S_{n,\alpha,\beta}^{q}(f,x) - f(x)\right| \leq \bar{\omega}_{\alpha}(f,x)\left[S_{n,\alpha,\beta}^{q}(|t-x|^{\alpha}, x)\right]^{\alpha/2} \leq \bar{\omega}_{\alpha}(f,x)\delta_n^{\frac{\alpha}{2}}(x).$$

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