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EFFICIENT DOMINATOR COLORING IN GRAPHS

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ABSTRACT

In 2006, Gera introduced and studied the dominator chromatic number, [5]. A dominator coloring of a graph G is a proper coloring in which each vertex of G dominates all the vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. A k-dominator coloring of G is a proper coloring in which every vertex of G dominate at least k color classes. We call a k-dominator coloring as efficient if every vertex of G dominate exactly k color classes. The k-dominator chromatic number $\chi_{d,k}(G)$ is the minimum number of colors required for a k-dominator coloring of G. In this paper, we characterize graphs which admit efficient 2-dominator coloring when $\chi_{d,2}(G) = 2$ and $\chi_{d,2}(G) = 3$. Further we identified some classes of cycles and prism graphs which admit efficient 2-dominator coloring. Finally we initiate the study of efficient 3-dominator coloring.

KEYWORDS: Dominator Coloring, K-Dominator Chromatic Number, Efficient K-Dominator Coloring.

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Original Article

1. INTRODUCTION

Let $G(V, E)$ be a simple finite and undirected graph(without loops and parallel edges), where V is called as the vertex set of G and E is called as the edge set of G. Coloring of graph vertices and finding the domination number of graphs are old and growing major research areas in graph theory that have been well studied in [4, 6, 7].

A proper coloring of a graph $G = (V, E)$ is a mapping f from the vertex set V to the set of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a color class of G, and so a coloring is a partition of V into color classes. The smallest number k for which G admits a coloring with k colors is the chromatic number $\chi(G)$ of G. A vertex subset $S \subseteq V$, of vertices in G) is called a dominating set if every vertex $v \in V - S$ should be adjacent with at least one vertex of S [6]. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G and the corresponding dominating set is called a γ-set [6].Gera et al. [5] introduced the concept of dominator chromatic number, which combines the concept of domination and coloring. Also they studied about dominator colorings and safe clique partitions.

A dominator coloring of G is a proper coloring in which each vertex of G dominate all the vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G. In [10] A.Sangeetha Devi et al., introduced k-dominator coloring in graphs. Also A.Sangeetha Devi et al., studied 2-dominator coloring on circulant graphs and obtain several results on this parameter [10].

A k-dominator coloring of G is a proper coloring in which every vertex of G dominate at least k color classes. The k-dominator chromatic number $\chi_{d,k}(G)$ is the minimum number of colors required for a k-dominator coloring of G. We call a vertex v as uniquely colored vertex if v is colored with one color and no other vertex of the graph receive this color.Note that, if a graph G admits k-dominator coloring, then the minimum degree $\delta(G) \geq k - 1$(since a vertex may dominate itself when it is uniquely colored). In the case of 2-dominator coloring, G will not have isolated vertices.

In this paper we introduce the concept of efficient k-dominator coloring in simple graphs. We call a k-dominator coloring as efficient if every vertex of G dominate exactly k color classes. Not all graphs admit efficient k-dominator coloring. For example, the cycle C_3 does not admit efficient 2-dominator coloring.

In this paper, we characterize graphs which admit efficient 2-dominator coloring when $\chi_{d,2}(G) = 2$ and $\chi_{d,2}(G) = 3$. Further we identified some classes of cycles and prism graphs which admit efficient dominator coloring. Finally the study of efficient 3-dominator coloring is initiated.

2. EFFICIENT 2-DOMINATOR COLORING

In this section we find the necessary and sufficient condition for the existence of efficient 2-dominator coloring when $\chi_{d,2}(G) = 2$ and $\chi_{d,2}(G) = 3$.

Further it is proved that some classes of cycles and prism graphs admit efficient 2-dominator coloring. Note that for any graph of order $n(\geq 2)$, the inequality holds $2 \leq \chi_{d,2}(G) \leq n$. From the definition of complete graph and efficient 2-dominator coloring, the next theorem follows.

Theorem 2.1

Let G be a connected graph of order $n(\geq 2)$ and $\chi_{d,2}(G) = 2$. Then G admits efficient 2-dominator coloring if and only if G is isomorphic to K_2.

Proof

Suppose $\chi_{d,2}(G) = 2$. Let C_1, C_2 be the two color classes obtained from an efficient 2-dominator coloring of G. Let $v \in C_1$. The v should be adjacent with exactly two color classes. Suppose the number of elements in C_1 exceeds one then v cannot dominate the color class C_1 and hence the coloring is not a 2-dominator coloring. Thus C_1 and similarly C_2 has exactly one element each and hence G is isomorphic to K_2. The converse part is trivial.

Theorem 2.2

Let G be a connected graph of order $n(\geq 2)$ and $\chi_{d,2}(G) = 3$. Then G admits efficient 2-dominator coloring if and only if G is a complete tripartite graph in which each partition contains at least two elements or G is a tripartite graph in which exactly one partition contains at least two elements and there is no edge between the one element partitions.

Proof

Let G be a connected graph of order $n(\geq 2)$ and $\chi_{d,2}(G) = 3$. Suppose G admits efficient 2-dominator coloring.

Let C_1, C_2 and C_3 be the three color classes obtained from an efficient 2-dominator coloring of G.

Case 1: Suppose each color class has only one element.

Let $C_1 = \{a\}$, $C_2 = \{b\}$ and $C_3 = \{c\}$. Now the vertex a dominate the color class C_1. Hence it must be adjacent with exactly one of the other two vertices to dominate another color class. Without loss of generality assume that a is adjacent with b. Thus b dominates the color classes C_1 and C_2. Hence both the vertices a and b will not be adjacent with c. In this case, the vertex c is isolated and hence it cannot dominate two color classes.

Case 2: Exactly one color class, say C_1 contains exactly one element a. Then a dominate C_1 and hence it will dominate exactly one of the remaining two color classes and without loss of generality, let it be C_2. Let $u, v \in C_2$ and $u \neq v$ (this is possible since C_2 has more than one element). Now both u and v have dominated the color class C_1 and these two vertices will not dominate the color class C_2. Thus u and v must be adjacent with all the vertices of C_3. Let $x \in C_3$. Now the vertex x will not dominate the color class C_3 (since C_3 has more than one element). Thus all the vertices of C_3 must dominate C_1. Hence, the vertex a will dominate all the three color classes, a contradiction to efficient 2-dominator coloring.

Case 3: Suppose exactly two color classes are singleton, say $C_1 = \{a\}$ and $C_2 = \{b\}$ and $|C_3| \geq 2$.

Let $u \in C_3$. Then u cannot dominate the color class C_3. Hence u will be adjacent with a and b and hence all the vertices of C_3 are adjacent with both a and b. In this case, a and b are not adjacent, otherwise both a and b will dominate all the three color classes, a contradiction.

Case 4: Suppose G is a tripartite graph in which every partition contains at least two elements.

Let $v \in C_1$. Then v must dominate exactly two color classes. Clearly v cannot dominate C_1 since $|C_1| \geq 2$. Thus v will dominate the color classes C_2 and C_3. Thus the vertex $v \in C_1$ is adjacent with all the vertices of C_2 and C_3. Similarly, we can say every vertex of one partition is adjacent with all the vertices of other two partitions. Thus G must be a complete tripartite graph. From the above four cases, G is a tripartite graph in which every partition contains at least two elements or G is a tripartite graph in which exactly one partition contains at least two elements and there is no edge between the one element partitions.

Conversely suppose G is a complete tripartite graph in which every partition contains at least two elements or G is a tripartite graph in which exactly one partition contains at least two elements and there is no edge between the one element partitions. In both the cases, by coloring the three partite sets with three different colors, we can prove that G admits efficient 2-dominator coloring.

Theorem 2.3

Let T be a tree of order $n(\geq 2)$. Then T admits efficient 2-dominator coloring if, and only if, $T = K_2$.

Proof

Suppose T admits efficient 2-dominator coloring. Suppose there exists a path of length three in T, say a, b, c. Since G is colored with two colors, say c_1, c_2, let us assume $c(b) = c_2$. Hence $c(a) = c(c) = c_1$. In this case the vertex a cannot dominate the color class C_1. Thus there exist no path of length 3 and hence $T = K_2$.Converse part is trivial.

For a cycle C_n, since every vertex has degree 2, we have $\chi_{d,3}(C_n) = n$ and in that case, the 3-dominator coloring number is n. Here we give some classes of cycles which admit efficient 2-dominator coloring.

Theorem 2.4

If $n(\geq 6)$ is a multiple of three, then C_n admits efficient 2-dominator coloring.

Proof

Let $V(C_n) = \{1, 2, 3, . . . , 3x\}$ for some integer $x \geq 2$. Let us color the vertices as follows:

We use one color to color all the vertices $3, 6, . . . , 3x$; and the other vertices are uniquely colored.

Clearly, in the above coloring we used $2x+ 1$ colors. Also the coloring is proper since we use same color only for the set of nonadjacent vertices $3, 6, . . . , 3x$. Further, except the vertices which are multiple of 3, all the other vertices are uniquely colored and hence each such vertex itself is a color class.

Now we prove that the above coloring is an efficient 2-dominator coloring. Let $v \in V(C_n)$.

Case 1: If v is a multiple of 3 and $v \neq 3x$.

In this case, the vertex v dominate only the two color classes $\{v - 1\}$ and $\{v + 1\}$.

Case 2: If $v = 3x$.

In this case, the vertex v dominate only the two color classes $\{v - 1\}$ and $\{1\}$. Case 3: If $v = 3i + 1$ for some integer i with $i = 0,1, . . , x - 1$.

In this case the vertex v dominate only the two color classes $\{v\}$ and $\{v + 1\}$. Case 4: If $v = 3i + 2$ for some integer i with $i = 0,1, . . , x - 1$.

In this case the vertex v dominate only the two color classes $\{v\}$ and $\{v - 1\}$. Hence C_n admits efficient 2-dominator coloring.

Theorem 2.5

Let $n(\geq 5)$ be an integer which is a multiple of five, then C_n admits efficient 2-dominator coloring.

Proof

Let $V(C_n) = \{1, 2, 3, . . . , 5x\}$ for some integer $x \geq 1$. Let us color the vertices as follows:

Let $v \in V(G)$. Then $v = 5i + j$ for some $0 \leq i \leq x - 1$ and $1 \leq j \leq 5$. For all i with $0 \leq i \leq x - 1$ and for $j = 1, 2$ and 4, the vertices are uniquely colored; and $c(5i + 3) = c(5(i + 1)) = i$. Clearly, in the above coloring we used $4x$ colors. Also the coloring is proper since we use same color only for the pairs of non adjacent vertices 3 and 5; 8 and 10; 13 and 15; . . .; $5(x-1)+3$ and $5x$. Further, except the vertices which are not of the form $5i + 3$ and $5i$, all the other vertices are uniquely colored and hence each such vertex itself is a color class.

Now we prove that the above coloring is an efficient 2-dominator coloring. Let $v \in V(C_n)$.

Case 1: If $v = 5i + 1$ for some integer i with $0 \leq i \leq x - 1$.

In this case, the vertex v dominate only the two color classes $\{v\}$ and $\{v + 1\}$.

Case 2: If $v = 5i + 2$ for some integer i with $0 \leq i \leq x - 1$.

In this case, the vertex v dominate only the two color classes $\{v - 1\}$ and $\{v\}$.

Case 3: If $v = 5i + 3$ for some integer i with $0 \leq i \leq x - 1$.

In this case, the vertex v dominate only the two color classes $\{v - 1\}$ and $\{v + 1\}$.

Case 4: If $v = 5i + 4$ for some integer i with $0 \leq i \leq x - 1$.

In this case, the vertex v dominate only the two color classes $\{v\}$ and $\{v - 1, v + 1\}$

Case 5: If $v = 5i$ for some integer i with $1 \leq i \leq x - 1$.

In this case, the vertex v dominate only the two color classes $\{v - 1\}$ and $\{v + 1\}$.

In the case of $v = 5x$, the vertex v dominate only the two color classes, namely $\{v - 1\}$ and $\{1\}$.

Hence C_n admits efficient 2-dominator coloring.

From the above two Theorems 2.4,2.5, we give the following conjecture:

Conjecture

Let $n(\geq 5)$ be an integer. Then C_n admits efficient 2-dominator coloring if, and only if, n is either a multiple of 3 or a multiple of 5.

The Cartesian product $G \times H$ of two graphs G and H, is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in E(G)$ with $y_1 = y_2$ or $(y_1, y_2) \in E(H)$ with $x_1 = x_2\}$ [6]. The Cartesian product of the cycle C_n with the complete graph K_2 is called prism graph which has $2n$ vertices and $3n$ edges. In the next lemma we give a family of prism graphs which admit efficient 2-dominator coloring.

Theorem 2.6

Let $n(\geq 6)$ be an even integer. Then the prism $G = C_n \times K_2$ admits efficient 2-dominator coloring.

Proof

Let the vertex set of G be $\{a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n\}$ and $E(G) = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_i : 1 \leq i \leq n - 1\} \cup \{a_n a_1, b_n b_1, \ldots, a_n b_n\}$.

Let us color the vertices as follows:

When i is odd, we take the vertices a_i and b_i as uniquely colored vertices. When i is even, we color all the vertices a_i with one color and all the vertices b_i with another color. Clearly, in the above coloring we used $n + 2$ colors.

Also the coloring is proper since we use same color only for the set of nonadjacent vertices $\{a_2, a_4, \ldots, a_n\}$ and for another set of nonadjacent vertices $\{b_2, b_4, \ldots, b_n\}$. Now we prove that the above coloring is an efficient 2-dominator coloring.

Let $v \in V(C_n)$.

Case 1: If $v = a_i$ and i is even.

In this case, the vertex v dominate only the two color classes, namely $\{a_{i-1}\},\{a_{i+1}\}$.

Case 2: If $v = a_i$ and i is odd.

In this case, the vertex v dominates only the two color classes, namely $\{a_i\}$ and $\{b_i\}$.

Similarly we can prove that each b_i dominate exactly two color classes. Hence G admits efficient 2-dominator coloring.

From the above Theorem 2.6 we conjecture the following:

Conjecture

Let $n(\geq 5)$ be an integer. Then $C_n \times K_2$ admits efficient 2- dominator coloring if, and only if, n is even.

3. EFFICIENT 3-DOMINATOR COLORING

In this section, we study the efficient 3-dominator coloring of graphs. Note that for any graph of order $n(\geq 3)$, we have $3 \leq \chi_{d,3}(G) \leq n$.

Theorem 3.1

Let G be a connected graph of order $n(\geq 3)$ and $\chi_{d,3}(\mathrm{G}) = 3$. Then G admits efficient 3-dominator coloring if and only if G is isomorphic to K_3.

Proof

Suppose $\chi_{d,3}(G) = 3$. Let C_1, C_2 and C_3 be the three color classes obtained from an efficient 3-dominator coloring of G. Let $v \in C_1$. The v should be adjacent with exactly three color classes. Suppose the number of elements in C_1 exceeds one then v cannot dominate the color class C_1 and hence the coloring is not a 3-dominator coloring. Thus $|C_1| = 1$. Similarly, we can prove $|C_2| = |C_3| = 1$. Hence G is isomorphic to K_3.The converse part is trivial.

Theorem 3.2

Let G be a connected graph of order $n(\geq 4)$, $\chi_{d,3}(G) = 4$ and G admits efficient 3-dominator coloring. If G is a 4-partite graph in which each partition contains more than one element, then it is a complete 4-partite graph.

Proof

Let C_1, C_2, C_3 and C_4 be the four color classes obtained from an efficient 3-dominator coloring of G. Let $v \in C_1$. Since $|C_1| \geq 2$, v cannot dominate the class C_1. Hence v must be adjacent with all the other three color classes. Similarly we can show that every vertex is adjacent with all the vertices of other color classes and hence G is complete 4-partite graph.

CONCLUSIONS

In this paper the results of some characterized graphs are found using the concept of efficient dominator coloring. This will be an initial study of extension of efficient k-dominator coloring.

REFERENCES

1. S. *Arumugam, Jay Bagya (Ed.),, 2012 On Dominator Colorings in Graphs." Vol. 122, 4561–4571, Proc. Indian Acad. Sci. (Math. Sci.),.*
2. *S.Chandrkumar, Ph.D thesis, 2015. A study on b coloring of graphs Manonmaniam Sundaranar University , , Tirunelveli.*
3. *T. Tamizh Chelvam and I. Rani. 2007. Dominating sets in Cayley graphs on Z_n, , Vol.**4.341**-345,Tamkang Journal of Mathematics .*
4. *Chartrand & Ping Zhang. (2009). Chromatic Graph Theory. New York:CRC Press.*
5. *R. Gera, C. Rasmussen (Ed.) 2006. Dominator colorings and safe clique partitions,Vol. 181, 19-32, Congressus Numerantium*
6. *T.W. Haynes, S.T. Hedetniemi (Ed.), 1998. Fundamentals of Domination in Graphs, Marcel Dekker Inc.,.*

7. *T.W. Haynes, S.T. Hedetniemi (Ed.), 1998. Domination in Graphs - Advanced Topics, Marcel Dekker Inc.*

8. *I.J. Dejter and O.Serra, 2003. Efficient dominating sets in Cayley graphs, Vol 129. 319-328, Discrete Appl. Math.,*

9. *Jia Huang, Jun-Ming Xu, 2008. The bondage numbers and efficient dominations of vertex-transitive graphs, Vol. 308, 571-582, Discrete Mathematics.*

10. *A.Sangeetha Devi , M.M.Shanmugapriya, 2016. 2-Dominator Coloring In Graphs, pp 43059-43062, International journal of Applied Engineering Research J. Graph Theory.*

11. *J. Lee, 2001. Independent perfect domination sets in Cayley graphs, 231-219. J. Graph Theory.*

12. *N. Obradovi´c, J. Peters (Ed.), 2007 .Efficient domination in circulant graphs with two chord lengths,253-258.Information Processing Letter.*