NUMERICAL SOLUTIONS OF EULER EQUATIONS BY RUNGE-KUTTA
DISCONTINUOUS GALERKIN METHOD

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ABSTRACT

Runge-Kutta discontinuous Galerkin (RKDG) method is a high order finite element method for solving hyperbolic conservation laws employing useful features from high resolution finite volume schemes, such as the exact or approximate Riemann solvers serving as numerical fluxes, TVD Runge-Kutta time discretizations and limiters. In most of the RKDG papers in the literature, the Lax-Friedrichs numerical flux is used due to its simplicity, although there are many other numerical fluxes which could also be used. This paper presents a Runge-Kutta discontinuous Galerkin (RKDG) method for the Euler equations of gas dynamics from the viewpoint of kinetic theory. The aim is also to define a way of taking into account high-order space discretization in limiting process, to make use of all the expansion terms of the approximate solution. A limiting procedure is carefully designed to suppress numerical oscillations. It is demonstrated by the numerical experiments that the proposed RKDG methods give higher resolution in solving problems with smooth solutions. Moreover, shock and contact discontinuities can be well captured by using the proposed methods.

KEYWORDS: Euler Equations, Exact Riemann Solver, High Order Accuracy, Limiters, Numerical Flux, TVD Scheme

INTRODUCTION

In the absence of diffusive phenomena due to viscous stresses, a 1-D compressible flow problem is described by the following hyperbolic equation set, namely the Euler equations [8], representing the conservation of mass, momentum, and energy:

\[ \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \]  

where the \( U \) and \( F \) are vectors defined as

\[ U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ E + \frac{2}{3} \rho u^2 \end{bmatrix} \]  

In the above equation, \( \rho \) is the density, \( u \) is the momentum, \( E \) is the energy, \( p \) is the pressure, and \( h \) is the dynamic enthalpy. This last variable is related to the other quantities by the following relation:

\[ h = E + \frac{p}{\gamma} \quad \text{with} \quad E = \frac{\rho u^2}{2} + \frac{\rho c_s^2}{\gamma - 1} \]  

with \( E \) being the internal energy. This system of three differential equations in four independent variables \( (\rho, \rho u, E, h) \) is closed by the equation of state, which is derived from thermodynamic principles. If the gas is calorically perfect and polytropic, then the pressure is related to the other variables by the relation,

\[ p = \frac{\rho c_s^2}{\gamma - 1} \]
where is the ratio of specific heats and takes the value of 1.4 for an ideal gas-air is often approximated as an ideal gas in compressible flow.

The first discontinuous Galerkin (DG) method was introduced by Reed and Hill (1973) in the framework of neutron transport (steady state linear hyperbolic equations). A major development of the DG method was carried out by Cockburn et al. (2001) to easily solve nonlinear time dependent hyperbolic conservation laws using explicit, nonlinearly stable high order Runge-Kutta time discretizations and DG discretization in space with exact or approximate Riemann solvers as interface fluxes. Jiang et al. (1994) proved that the DG method for the scalar case satisfies a local cell entropy inequality for the square entropy for arbitrary triangulations in any space dimension, and for any order of accuracy. Their results with shocks imply \( L^2 \) stability of the method for non-linear problems in the scalar case. Zhou et al. (2001) gave a numerical comparison of WENO finite volume and Runge-Kutta discontinuous Galerkin methods. Like typical finite element methods, the RKDG methods are better suited than finite difference methods to handle complicated geometries and arbitrary triangulations. Due to the discontinuous nature of the solution and the test function space, the solution is marched explicitly in time. Moreover, they can easily handle adaptivity strategies and parallel implementation due to their compactness and local communications. For these reasons, they have been widely used in practical applications. The purpose of this paper is to construct a quadrature-free RKDG method with higher-order accuracy for the Euler equations from the viewpoint of gas-kinetic theory. Like the traditional gas-kinetic schemes, the present RKDG method does not need the characteristic decomposition and the Riemann solver in computing the numerical flux at the surface of the finite elements. The integral term containing the non-linear flux can be computed exactly at the microscopic level. Moreover, a limiting procedure is carefully designed that preserves the accuracy in smooth regions and limits oscillations near the discontinuities. Some numerical experiments are conducted to validate the accuracy of the scheme and to demonstrate its ability in capturing discontinuities.

**RUNGE-KUTTA DISCONTINUOUS GALERKIN METHOD FOR THE EULER EQUATIONS**

In this section, we give the details of the procedure of the RKDG method. We start with the description in the one-dimensional case. However, we emphasize that the procedure described below does not depend on the specific basis chosen for the polynomials. We would like to solve the one-dimensional scalar conservation law

\[
\frac{\partial u}{\partial t} + f(u) = 0
\]

subject to the initial condition

\[
u(x, 0) = u_0(x)
\]

and periodic boundary conditions.

**ONE-DIMENSIONAL DG SPATIAL DISCRETIZATION**

This subsection is concerned with the DG spatial discretization for Euler equation [5] in one-space dimension. To discretize in space, we proceed as follows:

For each partition of the interval \((0, 1)\), \( \left\{ x_{j+\frac{1}{2}} \right\}_{j=0}^{N} \), we set \( I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \)

\( \Delta_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \) for \( f = 1, \ldots, N \) and denote the quantity by \( \Delta_x^N \). We seek an approximation \( u_{\Delta} \) to \( u \) such that for each time \( t \in [0, T] \), \( u_{\Delta}(x) \) belongs to the finite dimensional space
Numerical Solutions of Euler Equations by Runge-Kutta Discontinuous Galerkin Method

\[ V_h = V_h^k \equiv \{ v \in L^2(\Omega); v|_{I_j} \in P^k(I_j), \ j = 1, ..., N \} \]  

(7)

where \( P^k(I_j) \) denotes the space of polynomials in \( I \) of degree at most \( k \). In order to determine the approximate solution \( u_h \), we use a weak formulation that we obtain as follows. First, we multiply the equations (5) by arbitrary, smooth functions \( v \) and integrate over \( I_j \), and get, after a simple formal integration by parts,

\[ \int_{I_j} \partial_t u(x, t) v(x) \, dx - \int_{I_j} f(u(x, t)) \partial_x v(x) \, dx + \]

\[ f \left( u \left( x_{j+\frac{1}{2}}^{-} \right) \right) v \left( x_{j+\frac{1}{2}}^{-} \right) - f \left( u \left( x_{j+\frac{1}{2}}^{+} \right) \right) v \left( x_{j+\frac{1}{2}}^{+} \right) = 0 \]  

(8)

\[ \int_{I_j} u(x, t) v(x) \, dx = \int_{I_j} u_0(x) v(x) \, dx \]  

(9)

Next, we replace the smooth functions \( v \) by test functions \( v_h \) belonging to the finite element space \( V_h \) and the exact solution \( u \) by the approximate solution \( u_h \). Since the function \( u_h \) is discontinuous at the points \( x_{j+\frac{1}{2}} \), we must also replace the nonlinear flux numerical 'flux' that depends on the two values of \( u_h \) at the point \( (x_{j+\frac{1}{2}}, \xi) \) that is, by the function

\[ h(u_h)_{j+\frac{1}{2}}(\xi) = h \left( u \left( x_{j+\frac{1}{2}}^{-}, \xi \right), u \left( x_{j+\frac{1}{2}}^{+}, \xi \right) \right) \]  

(10)

that will be suitably chosen later. Note that we always use the same numerical flux regardless of the form of the finite element space. Thus, the approximate solution given by the DG-space discretization is defined as the solution of the following weak formulation:

\[ \forall j = 1, ..., N \forall v_h \in V_h \]  

\[ \int_{I_j} \partial_t u_h(x, t) v_h(x) \, dx - \int_{I_j} f(u_h(x, t)) \partial_x v_h(x) \, dx + \]

\[ h(u_h)_{j+\frac{1}{2}}(\xi) v_h \left( x_{j+\frac{1}{2}}^{-} \right) - h(u_h)_{j+\frac{1}{2}}(\xi) v_h \left( x_{j+\frac{1}{2}}^{+} \right) = 0 \]  

(11)

\[ \int_{I_j} u_h(x, t) v_h(x) \, dx = \int_{I_j} u_0(x) v_h(x) \, dx \]  

(12)

NUMERICAL FLUXES

To complete the definition of the approximate solution \( u_h \) it only remains to choose the numerical flux \( h \). To do that, we invoke our main point of view, namely, that we want to construct schemes that are perturbations of the so-called monotone schemes because monotone schemes, although only first-order accurate, are very stable and converge to the entropy solution. More precisely, we want that in the case \( k = 0 \), that is, when the approximate solution \( u_h \) is a piecewise-constant function, our DG-space discretization gives rise to a monotone scheme. Since in this case, for \( x \in I_j \) we can write

\[ u_h(x, t) = u_j^t, \]

we can rewrite our weak formulation (11), (12) as follows:
\[ \forall j = 1, \ldots, N: \]
\[ a_j u_j(x) + \frac{1}{h} \int_{x_j}^{x_{j+1}} u_j(x') dx' - a \left( u_{j+1}(x), u_j(x) \right) / \Delta x = 0, \tag{13} \]
\[ u_j(0) = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_0(x) dx \tag{14} \]

and it is well-known that this defines a monotone scheme if \( h(a, b) \) is a Lipschitz, consistent, monotone flux, that is, if it is,

1. locally Lipschitz and consistent with the flux \( f(u) \), i.e., \( h(a, u) = f(u) \).
2. a nondecreasing function of its first argument, and
3. a nonincreasing function of its second argument.

The best known examples of numerical flux satisfying the above properties is the Lax-Friedrichs flux [1]. The LF flux is one of the simplest and most widely used building blocks for the RKDG method and high order finite volume methods such as the ENO and WENO schemes [2]. However, the numerical viscosity of the LF flux is also the largest among monotone fluxes for scalar problems. The LF or the LLF flux is defined by

\[ h^L(u^-, u^+) = \frac{1}{2} \left[ f(u^-) + f(u^+) - \alpha (u^- - u^+) \right]. \tag{15} \]

where for the LF flux, is taken as an upper bound over the whole line for \( |f'(u)| \) in the scalar case, or for the absolute value of eigenvalues of the Jacobian for the system case, and for the LLF flux \( \alpha \) is taken as an upper bound between \( u^- \) and \( u^+ \). Taking advantage of the good property of Legendre Polynomial \( L \) take the basis in the following form:

\[ \phi_j(x) = \frac{1}{\sqrt{2}} \left( 2x - x_j \right) / \Delta x \]

where \( L(x) \) is the \( j^{th} \) Legendre polynomial. We express our approximate solution \( u_h \) as follows:

\[ u_h(x, t) = \sum_{j=0}^{N} u_j \phi_j(x) \tag{16} \]

By integrating by parts and equations (12)

\[ \int_{x_j}^{x_{j+1}} f(u_h) dx = 0 \tag{17} \]
\[ \int_{x_j}^{x_{j+1}} u_h(0) \phi_j(x) dx = \int_{x_j}^{x_{j+1}} u_0(x) \phi_j(x) dx \tag{18} \]

can be converted into following simplified form

\[ \left( \frac{1}{\sqrt{2} x_j \Delta x^2} \int_{x_j}^{x_{j+1}} a_j u_j(x) dx - \frac{1}{\Delta y} \int_{x_j}^{x_{j+1}} f(u_h(x, \xi)) \phi_j(x) dx + \frac{1}{\Delta y} \int_{x_j}^{x_{j+1}} h(u_h(\xi, x)) \phi_j(x) dx \right) = 0 \tag{19} \]
\[ u_j(x) \phi_j(x) dx \tag{20} \]

This shows that after discretizing in space the problem (5) by the DG method, we obtain a system of ODEs for the degrees of freedom that we can rewrite as follows:

\[ \frac{d}{dt} u_h = L_h(u_h) \tag{21} \]
The element \( L_k(u_0) \) of \( V_h \) is, of course, the approximation to \(-f(\mathbf{u})_x\) provided by the DG space discretization. If we choose a different local basis, the local mass matrix could be a full matrix but it will always be a matrix of order \( J + 1 \).

By inverting it by means of a symbolic manipulator, we can always write the equations for the degrees of freedom of \( u_0 \) as an ODE system of the form above.

**TIME DISCRETIZATIONS**

Total Variation Diminishing Runge-Kutta Schemes is used to solve a system of ODEs:

\[
\mathbf{u}_k = L(\mathbf{u})
\]

with suitable initial conditions, resulting from a method of lines approximation to a hyperbolic conservation law:

\[
\mathbf{u}_k = -f(\mathbf{u})_x
\]

where the spatial derivative \( f(\mathbf{u})_x \) is approximated by a discontinuous finite element approximation denoted by \(-L(\mathbf{u})\) which has the property that the total variation of the numerical solution

\[
TV(\mathbf{u}) = \sum_j |u_{j+1} - u_j|
\]

does not increase \( TV(\mathbf{u}^{n+1}) \leq TV(\mathbf{u}^n) \) for a first order in time Euler forward stepping for a first order in time Euler forward stepping

\[
\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \mathbf{L}(\mathbf{u}^n)
\]

under suitable restriction on \( \Delta t \) i.e

\[
\Delta t \leq \Delta t_1
\]

The objective of the high order TVD Runge-Kutta time discretization \[2\], is to maintain the TVD property (26) while achieving higher order accuracy in time, perhaps with a different time step restriction than (28):

\[
\Delta t \leq \frac{c\Delta t_1}{2}
\]

In this paper, we will use a third-order TVD Runge-Kutta scheme of C.-W. Shu et al. (1988) when solving a non-smooth problem and a fourth-order non-TVD Runge-Kutta scheme for solving smooth problems. More precisely, if we rewrite ODE system (21)–(23) in a compact form: \( \mathbf{U}^{(t)} = L(\mathbf{U}) \), then the third-order TVD Runge-Kutta scheme is of the form

\[
\mathbf{U}^{(1)} = \mathbf{U}^n + \frac{1}{3} \Delta t \mathbf{L}(\mathbf{U}^{\frac{1}{3}})
\]

\[
\mathbf{U}^{(2)} = \mathbf{U}^n + \frac{2}{3} \Delta t \mathbf{L}(\mathbf{U}^{\frac{2}{3}})
\]

\[
\mathbf{U}^{(3)} = \mathbf{U}^n + \frac{3}{2} \Delta t \mathbf{L}(\mathbf{U}^{\frac{3}{2}})
\]

While the fourth-order non-TVD Runge-Kutta scheme[2] is of the form

\[
\mathbf{U}^{(1)} = \mathbf{U}^n + \frac{1}{4} \Delta t \mathbf{L}(\mathbf{U}^{\frac{1}{4}})
\]

\[
\mathbf{U}^{(2)} = \mathbf{U}^n + \frac{1}{2} \Delta t \mathbf{L}(\mathbf{U}^{\frac{1}{2}})
\]

\[
\mathbf{U}^{(3)} = \mathbf{U}^n + \frac{3}{4} \Delta t \mathbf{L}(\mathbf{U}^{\frac{3}{4}})
\]
The numerical results show the accuracy of the schemes for the one-dimensional system. The fourth-order Runge-Kutta method (31) will be employed. It is known that if the solution discontinuities are strong then the RKDG methods will generate significant oscillations and even non-linear instability. To fix this, typically a local projection or non-linear limiter should be used after each Runge–Kutta inner stage or after some complete Runge-Kutta steps.

**Example 1:** This example is to solve Sod’s shock tube problem. The domain is taken as $[0, 1]$ which is divided into 100 cells. The initial data are

$$\begin{align*}
\rho(x,0) &= \begin{cases}
1 & x < 0.5 \\
0.125 & x \geq 0.5
\end{cases}, \\
\rho u(x,0) &= \begin{cases}
0 & x < 0.5 \\
0.625 & x \geq 0.5
\end{cases}, \\
\rho e(x,0) &= \begin{cases}
1 & x < 0.5 \\
0.4 & x \geq 0.5
\end{cases}
\end{align*}$$

**Figure 1:** The Density at $\tau = 0.2$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells

**Figure 2:** The Velocity at $\tau = 0.2$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells
We output the solutions at $\tau = 0.2$, which consists of a left rarefaction wave, a contact discontinuity, and a right moving weak shock. We got results after 68 iterations. Fig. 1 shows the density, the pressure and the velocity obtained by the RKDG method with using a limiting procedure. It is seen that the shock is resolved almost exactly by our RKDG method and the resolution of the contact discontinuous is improved as $q$ is increased, although numerical oscillation becomes serious in the same time. After a limiting procedure is used, the numerical oscillation can been essentially suppressed while the shock is still well-resolved, although the resolution of the contact waves is slightly decreased.

**Example 2:** This example is to solve 123 problem. The domain is taken as $[0, 1]$ which is divided into 100 cells. The initial data are

$$(\rho, u, p) = \begin{cases} (1, -2.0, 0.4), & \tau \leq 0.4 \\ (1.5, 2.0, 0.4), & \tau > 0.4 \end{cases}$$
Figure 5: The Velocity at $t = 0.15$ is Obtained by $P^1$ based RKDG Method with 100 Grid Cells

Figure 6: The Pressure at $t = 0.15$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells

We output the solutions at $t = 0.15$, which consists of a left rarefaction wave, a contact discontinuity, and a right moving weak shock. We got results after 52 iterations. Figs. 4-6 show the density, the pressure and the velocity obtained by the RKDG method with using a limiting procedure.

Example 3: This example is to solve the left half of the blast wave problem of Woodward & Colella. The domain is taken as $[0,1]$ which is divided into 100 cells. The initial data are

$$(\rho_0, u_0, p) = \begin{cases} (1,0,1000.0) & x \leq 0.4 \\ (1.0,0.0,0.01) & x \geq 0.4 \end{cases}$$
Figure 7: The Density at $t = 0.012$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells

Figure 8: The Velocity at $t = 0.012$ is Obtained by $P^1$ based RKDG Method with 100 Grid Cells

Figure 9: The Pressure at $t = 0.012$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells

We output the solutions at $t = 0.012$, which consists of a left rarefaction wave, a contact discontinuity, and a right moving weak shock. We got results after 76 iterations. Figs. 7-9 show the density, the pressure and the velocity obtained by the RKDG method with using a limiting procedure.
**Example 4:** This example is to solve right half of the Woodward & Colella problem. The domain is taken as $[0, 1]$ which is divided into 100 cells. The initial data are

$$\rho(x,0,p) = \begin{cases} (1.0, 0.0, 0.0) & x \leq 0.4 \\ (2.0, 0.0, 200.0) & x \geq 0.4 \end{cases}$$

![Figure 10: The Density at $t = 0.035$ is Obtained by $P^1$ based RKDG Method with 100 Grid Cells](image)

We output the solutions at $t = 0.035$, which consists of a left rarefaction wave, a contact discontinuity, and a right moving weak shock. We got results after 132 iterations. Figs. 10-12 show the density, the pressure and the velocity obtained by the RKDG method with using a limiting procedure.
**Example 5:** This example is to solve Stationary contact discontinuities problem. The domain is taken as $[0, 1]$ which is divided into 100 cells. The initial data are

$$(\rho, u, p) = \begin{cases} 
(5.99925, 19.5975, 400.694), & x \leq 0.4 \\
(5.99242, -5.19533, 38.0959), & x \geq 0.4
\end{cases}$$

Figure 13: The Density at $t = 0.012$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells

Figure 14: The Velocity at $t = 0.012$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells

Figure 15: The Pressure at $t = 0.012$ is Obtained by $P^1$ Based RKDG Method with 100 Grid Cells
We output the solutions at $t = 0.012$, which consists of a left rarefaction wave, a contact discontinuity, and a right moving weak shock. We got results after 86 iterations. Figs. 13-15 show the density, the pressure and the velocity obtained by the RKDG method with using a limiting procedure.

CONCLUSIONS

In this paper, we have presented a Runge-Kutta discontinuous Galerkin (RKDG) method for the Euler equations of gas dynamics from the viewpoint of gas-kinetic theory. Like the traditional gas-kinetic schemes, the proposed RKDG method is characteristic decomposition free and Riemann solver free since the integral term containing the non-linear flux can be computed exactly at the microscopic level. When the solutions to the Euler equations are either smooth or have weak discontinuities, they can be resolved by our RKDG methods very efficiently without using any limiting procedure. However, if the discontinuities are strong then the RKDG methods will generate significant oscillations and even non-linear instability. A limiting procedure is added in our RKDG methods to fix this problem. The accuracy and efficiency of the proposed methods are demonstrated by several numerical examples.

REFERENCES


