ON THE NUMERICAL SOLUTION FOR THE LINEAR FRACTIONAL KLIEN-GORDON EQUATION USING LEGENDRE PSEUDOSPECTRAL METHOD

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ABSTRACT

Fractional differential equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering and others. However, many researchers remain unaware of this field. In this paper, an efficient numerical method for solving the linear fractional Klein-Gordon equation is considered. The fractional derivative is described in the Caputo sense. The method is based on Legendre approximations. The properties of Legendre polynomials are utilized to reduce the proposed problem to a system of ordinary differential equations, which solved using the finite difference method. Numerical solutions are presented and the results are compared with the exact solution.

KEYWORDS: Fractional Klein-Gordon Equation; Caputo Fractional Derivative; Finite Difference Method; Legendre Polynomials

INTRODUCTION

Ordinary and partial fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering (Bagley & Torvik (1984)). Consequently, considerable attention has been given to the solutions of FDEs of physical interest. Most FDEs do not have exact solutions, so approximate and numerical techniques (He (1998), Sweilam et al. (2011), Sweilam et al. (2012a-e)), must be used. Recently, several numerical methods to solve FDEs have been given such as variational iteration method (Inc (2008)), homotopy perturbation method (El-Sayed et al. (2012a, b), Sweilam et al. (2007)), Adomian decomposition method (El-Sayed (2003), Jafari & Daftardar (2006)), homotopy analysis method (Hashim et al. (2009)) and collocation method (Khader (2011), Khader (2012), Khader & Hendy (2012b), Sweilam & Khader (2010), Sweilam et al. (2012c-e)).

The linear or nonlinear PDEs arising in physics and engineering play an important role in mathematical modelling. The hyperbolic PDEs model the vibrations of structures. So, searching the numerical and exact solutions to these linear or nonlinear models gains importance. For this reason, many methods were developed for solving differential equations in literature. The Klein-Gordon equation plays a significant role in mathematical physics and many scientific applications such as solid-state physics, nonlinear optics, and quantum field theory (Wazwaz (2006)). The equation has attracted much attention in studying solitons (Sassaman & Biswas (2011a, b)) and condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations (El-Sayed (2003)). Wazwaz has obtained the various exact travelling wave solutions such as compactons, solitons and periodic solutions by using the tanh method (Wazwaz (2006)). The study of numerical solutions of the Klein-Gordon equation has been investigated considerably in the last few years. In the previous studies, the most papers have carried out different spatial discretization of the equation (Golmankhaneh et al. (2011), Yusufoglu (2008)). Where, the numerical solution using radial basis functions is given in (Dehghan & Shokri (2009)), collocation and finite
difference-collocation methods for the solution of proposed problem is introduced in (Lakestani & Dehghan (2010)), finally, the tension spline approach for the numerical solution of nonlinear Klein-Gordon equation is implemented in (Rashidini & Mohammadi (2010)), (Khader et al. (2011)) introduced a new approximate formula of the fractional derivative and used it to solve numerically the fractional diffusion equation. Also, (Khader and Hendy (2012a)) used this formula to solve numerically the fractional delay differential equations.

We present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

**DEFINITION**

The Caputo fractional derivative operator $D^\alpha$ of order $\alpha$ is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+m}} \, dt, \quad \alpha > 0, \ x > 0,$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x),$$

where $\lambda$ and $\mu$ are constants.

For the Caputo's derivative we have (Podlubny (1999))

$$D^\alpha C = 0, \quad C \text{ is a constant},$$

$$D^\alpha x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < [\alpha]; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq [\alpha]. \end{cases}$$

We use the ceiling function $[\alpha]$ to denote the smallest integer greater than or equal to $\alpha$ and $\mathbb{N}_0 = \{0,1,2,...\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see (Das (2008), Podlubny (1999)).

Our main goal in this paper is concerned with the application of Legendre pseudospectral method to obtain the numerical solution of the linear fractional Klein-Gordon equation (LFKGE) of the form

$$u_{xx}(x,t) - d(x)D^\alpha u(x,t) = f(x,t), \quad 0 < x < 1,$$

the parameter $\alpha$ refers to the Caputo fractional order of spatial derivatives with $1 < \alpha \leq 2$. The function $f(x,t)$ is the source term. We also assume the following initial conditions

$$u(x,0) = g_1(x), \quad u_x(x,0) = g_2(x), \quad x \in (0,1),$$

and the Dirichlet boundary conditions

$$u(0,t) = h_1(t), \quad u(1,t) = h_2(t),$$

where $g_1, g_2, h_1$ and $h_2$ are known functions and $u(x,t)$ is the unknown function.

Note that at $\alpha = 2$, Eq.(3) is the classical linear Klein-Gordon equation

$$u_{xx}(x,t) - d(x)u_{xx}(x,t) = f(x,t).$$

Our idea is to apply the Legendre collocation method to discretize (3) to get a linear system of ODEs thus greatly
simplifying the problem, and use finite difference method (Meerschaert & Tadjeran (2006), Smith (1965)) to solve the resulting system.

Legendre polynomials are well known family of orthogonal polynomials on the interval $[-1,1]$ that have many applications (Bell (1968), Khader et al. (2011), Khader & Hendy (2012a)). They are widely used because of their good properties in the approximation of functions. However, with our best knowledge, very little work was done to adapt these polynomials to the solution of FDEs.

The organization of this paper is as follows. In the next section, the approximation of fractional derivative $D^\alpha y(x)$ is obtained. Section 3 summarizes the application of Legendre collocation method to the solution of (3). As a result a system of ordinary differential equations is formed and the solution of the considered problem is introduced. In section 4, some numerical results are given to clarify the proposed method. Also a conclusion is given in section 5.

**AN APPROXIMATE FORMULA OF THE FRACTIONAL DERIVATIVE**

The well known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula (Bell (1968))

$$L_{k+1}(z) = \frac{2k+1}{k+1} z L_k(z) - \frac{k}{k+1} L_{k-1}(z), \quad k = 1, 2, \ldots,$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $[0,1]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2x - 1$.

Let the shifted Legendre polynomials $L_k(2x - 1)$ be denoted by $P_k(x)$. Then $P_k(x)$ can be obtained as follows

$$P_{k+1}(x) = \frac{(2k+1)(2x-1)}{(k+1)} P_k(x) - \frac{k}{k+1} P_{k-1}(x), \quad k = 1, 2, \ldots,$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The analytic form of the shifted Legendre polynomials $P_k(x)$ of degree $k$ is given by

$$P_k(x) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k + i)!}{(k - i)(i)!} x^i.$$

Note that $P_k(0) = (-1)^k$ and $P_k(1) = 1$. The orthogonality condition is

$$\int_0^1 P_i(x) P_j(x) \, dx = \begin{cases} \frac{1}{2i + 1}, & \text{for } i = j; \\ 0, & \text{for } i \neq j \end{cases}$$

The function $y(x)$, which is square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{i=0}^{\infty} y_i P_i(x),$$

where the coefficients $y_i$ are given by $y_i = (2i + 1) \int_0^1 y(x) P_i(x) \, dx$, $i = 1, 2, \ldots$

In practice, only the first $(m+1)$-terms of shifted Legendre polynomials are considered. Then we have

$$y_m(x) = \sum_{i=0}^{m} y_i P_i(x).$$
The main approximate formula of the fractional derivative is given in the following theorem.

THEOREM

Let \( y(x) \) be approximated by shifted Legendre polynomials as (9) and also suppose \( \alpha > 0 \) then

\[
D^\alpha (y_m(x)) = \sum_{i=0}^{m} \sum_{k=0}^{i} y_i \, w_{i,k} \, x^{k-\alpha},
\]

where \( w_{i,k}^{(\alpha)} \) is given by

\[
w_{i,k}^{(\alpha)} = \frac{(-1)^i (i + k)!}{(i - k)! (k!)^2 \Gamma(k + 1 - \alpha)}. \tag{11}
\]

PROOF

Since the Caputo's fractional differentiation is a linear operation we have

\[
D^\alpha (y_m(x)) = \sum_{i=0}^{m} y_i \, D^\alpha (P_i(x)). \tag{12}
\]

Employing Eqs.(1)-(2) in Eq.(7) we have

\[
D^\alpha P_i(x) = 0, \quad i = 0, 1, \ldots, [\alpha] - 1, \quad \alpha > 0. \tag{13}
\]

Therefore, for \( i = [\alpha], [\alpha] + 1, \ldots, m \) and by using Eqs.(1)-(2) and (7) we get

\[
D^\alpha P_i(x) = \sum_{k=0}^{i} \frac{(-1)^i (i + k)!}{(i - k)! (k!)^2 \Gamma(k + 1 - \alpha)} \, D^\alpha (x^k) = \sum_{k=0}^{i} \frac{(-1)^i (i + k)!}{(i - k)! (k!)^2 \Gamma(k + 1 - \alpha)} \, x^{k-\alpha}, \tag{14}
\]

a combination of Eqs.(12), (13) and (14) leads to the desired result.

PROCEDURE SOLUTION OF THE LINEAR FRACTIONAL KLIEN-GORDON EQUATION

Consider the LFKGE of type given in Eq.(3). In order to use Legendre collocation method, we first approximate \( u(x,t) \) as

\[
u_m(x,t) = \sum_{i=0}^{m} u_i(t) \, P_i(x). \tag{15}
\]

From Eqs.(3), (15) and Theorem 1 we have

\[
\sum_{i=0}^{m} \frac{d^2 u_i(t)}{d t^2} \, P_i(x) - d(x) \sum_{i=0}^{m} \sum_{k=0}^{i} u_i(t) \, w_{i,k}^{(\alpha)} \, x^{k-\alpha} = f(x,t). \tag{16}
\]

We now collocate Eq.(16) at \((m + 1 - [\alpha])\) points \( x_p \), \( p = 0, 1, \ldots, m - [\alpha] \) as

\[
\sum_{i=0}^{m} \tilde{u}_i(t) \, P_i(x_p) - d(x_p) \sum_{i=0}^{m} \sum_{k=0}^{i} u_i(t) \, w_{i,k}^{(\alpha)} \, x_p^{k-\alpha} = f(x_p,t). \tag{17}
\]

For suitable collocation points we use roots of shifted Legendre polynomial \( P_{m+1-[\alpha]}(x) \).

Also, by substituting Eq.(15) in the boundary conditions (5) we can obtain \([\alpha]\) equations as follows

\[
\sum_{j=0}^{m} (-1)^j u_j(t) = h_j(t), \quad \sum_{j=0}^{m} u_j(t) = h_z(t). \tag{18}
\]
Eq.(17), together with \[[\alpha]\] equations of the boundary conditions (18), give \((m + 1)\) of ordinary differential equations which can be solved, for the unknowns \(u_i, i = 0, 1, \ldots, m\), using the finite difference method, as described in the following section.

**NUMERICAL RESULTS**

In this section, we implement the proposed method to solve LFKGE (3) with \(\alpha = 1.8\), of the form

\[ u_s(x, t) - d(x)D^{1.8}u(x, t) = f(x, t), \quad 0 < x < 1, \quad t > 0, \]

with the coefficient function: \(d(x) = \Gamma(1.2)x^{1.8}\),

the source function: \(f(x, t) = x^2(4x - 1)e^{-t}\),

under initial conditions \(u(x, 0) = x^2(1 - x), \quad u_t(x, 0) = x^2(x - 1)\),

and Dirichlet conditions \(u(0, t) = u(1, t) = 0\).

The exact solution to this problem is \(u(x, t) = x^2(1 - x)e^{-t}\), which can be verified by applying the fractional differential formula (2).

We apply the proposed method with \(m = 3\), and approximate the solution as follows

\[ u_3(x, t) = \sum_{i=0}^{3} u_i(t)P_i(x). \]  

Using Eq.(17) we have

\[ \sum_{i=0}^{3} u_i(t)P_i(x_p) - d(x_p)\sum_{i=2}^{m} u_i(t)w_{i,k}^{(1.8)}x_p^{k-1.8} = f(x_p, t), \quad p = 0, 1, \]

where \(x_p\) are roots of shifted Legendre polynomial \(P_2(x)\), i.e. \(x_0 = 0.211324, \quad x_1 = 0.788675\).

By using Eqs.(18) and (20) we can obtain the following system of ODEs

\[ \ddot{u}_0(t) + k_1\dot{u}_1(t) + k_2\dot{u}_3(t) - R_1u_2(t) - R_2u_4(t) = f_0(t), \]

\[ \ddot{u}_1(t) + k_1\dot{u}_1(t) + k_2\dot{u}_3(t) - R_{11}u_2(t) - R_{22}u_4(t) = f_1(t), \]

\[ u_0(t) - u_1(t) + u_2(t) - u_3(t) = 0, \]

\[ u_0(t) + u_1(t) + u_2(t) + u_3(t) = 0, \]

where

\[ k_1 = P_1(x_0), \quad k_2 = P_3(x_0), \quad k_{11} = P_1(x_1), \quad k_{22} = P_3(x_1), \]

\[ R_1 = d(x_0)w_{2,2}^{(1.8)}x_0^{0.2}, \quad R_2 = d(x_0)w_{3,2}^{(1.8)}x_0^{0.2} + w_{3,3}^{(1.8)}x_0^{1.2}, \]

\[ R_{11} = d(x_1)w_{2,2}^{(1.8)}x_1^{0.2}, \quad R_{22} = d(x_1)w_{3,2}^{(1.8)}x_1^{0.2} + w_{3,3}^{(1.8)}x_1^{1.2}. \]

Now, to use FDM for solving the system (21)-(24), we use the notations \(t_i = i\tau\) to be the integration time \(0 \leq t_i \leq T\), \(\tau = T/N\), for \(i = 0, 1, \ldots, N\) for an arbitrary integer number \(N\). Define \(u_i^n = u_i(t_n), f_i^n = f_i(t_n)\).
Then the system (21)-(24), is discretized in time and takes the following form

$$\begin{align*}
\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\tau^2} + k_1 u_1^{n+1} - 2u_1^n + u_1^{n-1} + k_2 u_2^{n+1} - 2u_2^n + u_2^{n-1} - R_1 u_3^{n+1} - R_2 u_3^{n-3} &= f_0^{n+1}, \\
\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\tau^2} + k_3 u_1^{n+1} - 2u_1^n + u_1^{n-1} + k_4 u_2^{n+1} - 2u_2^n + u_2^{n-1} - R_1 u_3^{n+1} - R_2 u_3^{n-3} &= f_1^{n+1}, \\
0 &= 0, \\
0 &= 0.
\end{align*}$$

We can write the above system (25)-(28) in the following matrix form

$$\begin{pmatrix}
1 & k_1 & \tau^2 R_1 & k_2 & \tau^2 R_2 \\
1 & k_3 & \tau^2 R_3 & k_4 & \tau^2 R_5 \\
1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
u_0^{n+1} \\
u_1^{n+1} \\
u_2^{n+1} \\
u_3^{n+1}
\end{pmatrix}
= \begin{pmatrix}
u_0^{n+1} \\
u_1^{n+1} \\
u_2^{n+1} \\
u_3^{n+1}
\end{pmatrix} + \begin{pmatrix}
u_0^{n+1} \\
u_1^{n+1} \\
u_2^{n+1} \\
u_3^{n+1}
\end{pmatrix}.$$

We use the notation for the above system

$$AU^{n+1} = BU^n - CU^{n-1} + F^{n+1}, \quad \text{or,} \quad U^{n+1} = A^{-1}BU^n - A^{-1}CU^{n-1} + A^{-1}F^{n+1},$$

where $U^n = (u_0^n, u_1^n, u_3^n, u_3^n)^T$ and $F^n = (\tau^2 f_0^n, \tau^2 f_1^n, 0, 0)^T.$

The obtained numerical results by means of the proposed method are shown in Table 1 and figures 1-2. In Table 1, the absolute errors between the exact solution $u_{\text{ex}}$ and the approximate solution $u_{\text{approx}}$ at $m = 3$ and $m = 5$ with the final time $T = 2$ are given. Also, in figures 1 and 2, comparison between the exact solution and the approximate solution at $T = 1$ with time step $\tau = 0.0025$, $m = 3$ and $m = 5$, respectively, are presented.

**Table 1: The Absolute Error Between the Exact Solution and the Approximate Solution at**

<table>
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<tr>
<th>$x$</th>
<th>$u_{\text{ex}} - u_{\text{approx}}$ at $m = 3$</th>
<th>$u_{\text{ex}} - u_{\text{approx}}$ at $m = 5$</th>
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</tr>
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CONCLUSIONS

In this paper, we considered the numerical solutions for linear fractional Klein-Gordon equation using the application of Legendre pseudo-spectral method. The properties of the Legendre polynomials are used to reduce the proposed problem to the solution of system of ordinary differential equations which solved by using FDM. The fractional derivative is considered in the Caputo sense. The solutions obtained using the suggested method are in excellent agreement with the exact solution and show that this approach can be solved the problem effectively. Although we only considered a model problem in this paper, the main idea and the used techniques in this work are also applicable to many other problems. It is evident that the overall errors can be made smaller by adding new terms from the series (15). Comparisons are made between the approximate solution and the exact solution to illustrate the validity and the great potential of the technique. All computations in this paper are done using Matlab 8.

REFERENCES


