LINE DOUBLE DOMINATION IN GRAPHS

M. H. MUDDEBIHAL¹ & SUHAS P. GADE²

¹Department of Mathematics Gulbarga University, Gulbarga, Karnataka, India
²Department of Mathematics, Sangameshwar College, Solapur, Maharashtra, India

ABSTRACT

Let \( G = (V, E) \) be a graph. A set \( D \subseteq V \) is called a dominating set if every vertex in \( V - D \) is adjacent to at least one vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a minimal dominating set. A subset \( D^d \) of \( V[L(G)] \) is a double dominating set of \( L(G) \) if for every vertex \( v \in V[L(G)] \), \( |N[v] \cap D^d| \geq 2 \), that is \( v \) is in \( D^d \) and has at least one neighbour in \( D^d \) or \( v \) is in \( V[L(G)] - D^d \) and has at least two neighbours in \( D^d \). The line double domination number \( \gamma_{d}d(G) \) is the minimum cardinality among all line double dominating sets of \( L(G) \). In this paper many bounds on \( \gamma_d d(G) \) were obtained in terms of vertices, edges and other different parameters of \( G \), but not the elements of \( L(G) \), further we develop its relationship with other different domination parameters.

KEYWORDS: Line Graph, Dominating Set, Double Dominating Set, Double Domination Number

Subject Classification Number: AMS-05C69, 05C70.

1. INTRODUCTION

All graphs under consideration are finite undirected and loop-less without multiple edges. Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). As usual \( p = |V| \) and \( q = |E| \) denote the number of vertices and edges of a graph \( G \) respectively. In general we use \( <X> \) to denote the sub-graph induced by the set of vertices \( X \) and \( N(v) \) and \( N[v] \) denote the open and closed neighbourhood of a vertex \( v \), respectively. The minimum (maximum) degree among the vertices of \( G \) is denoted by \( \delta(G), \Delta(G) \). A vertex of degree one is called an end vertex. Also \( \beta_0(G), \beta_1(G) \) is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of \( G \). \( \chi(G), \chi^*(G) \) is the minimum \( n \) for which \( G \) has an \( n \)-vertices (\( n \)-edges) colourings. A line graph \( L(G) \) is the graph whose vertices correspond to the edges of \( G \) and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) are adjacent. We begin with some standard definitions from domination theory. Let \( G = (V, E) \) be a graph. A set \( D \) of vertices in a graph \( G \) is called a dominating set of \( G \) if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The domination number of \( G \), denote by \( \gamma(G) \) is the minimum cardinality of a dominating set. A set \( D \) subset of \( V[L(G)] \) is said to be a dominating set of \( L(G) \), if every vertex not in \( D \) is adjacent to a vertex in \( D \) of \( L(G) \). The domination number of \( L(G) \) is denoted by \( \gamma[L(G)] \) is the minimum cardinality of a dominating set. A set \( D^d \) subset of \( V[L(G)] \) is called a double dominating set of a \( L(G) \) if
every vertex in $V[L(G)]$ is dominated by at least two vertices in $S$. Or a subset $D^d$ of $V[L(G)]$ is a double dominating set of $L(G)$ if for every vertex $v \in V[L(G)]$, $|N[v] \cap D^d| \geq 2$, that is $v$ is in $D^d$ and has at least one neighbour in $D^d$ or $v$ is in $V[L(G)] - D^d$ has at least two neighbours in $D^d$ and is denoted by $\gamma_{ddl}(G)$. In this paper, many bounds on $\gamma_{ddl}(G)$ were obtained in terms of vertices, edges of $G$ but not the member of $L(G)$. Also we establish line double domination of a line graph and express the results with other different domination parameters of $G$.

We need the following Theorem to prove our further results.

**Theorem A[1]**: Let $G$ be a graph with $diam(G) = 2$ then $\gamma_v(G) \leq \Delta(G) + 1$.

**Theorem B[4]**: If $G$ is a graph without isolated vertices and $p \geq 3$ then $\gamma_{ss}(G) = \alpha(G)$.

**Theorem C[4]**: A non split dominating set $D$ of $G$ is minimal if and only if for each vertex $v \in D$ there exist a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$.

**Theorem D[2]**: For any connected $(p, q)$ graph $G$, $\chi(G) \leq \Delta(G) + 1$.

**Theorem E[3]**: For any connected $(p, q)$ graph $G$, $\frac{diam(G)}{3} \leq \gamma(G)$.

**Observation 1**: For any connected $(p, q)$ graph $G$, $p - \gamma_{ddl}(G) \leq 1$.

2. Upper Bound for $\gamma_{ddl}(G)$:

We shall establish the upper bound for $\gamma_{ddl}(G)$ in terms of the vertices of $G$.

**Theorem 1**: For any connected $(p, q)$ graph $G$, $\gamma_{ddl}(G) \leq p - 1$. Equality holds for $P_3, C_3, C_4, C_5$.

**Proof**: Suppose $D^d$ is a double dominating set of $L(G)$. Then by definition of double domination, $|V[L(G)]| \geq 2$. Further by observation, $p - \gamma_{ddl}(G) \geq 1$. Clearly it follows that $\gamma_{ddl}(G) \leq p - 1$. Suppose $G$ is isomorphic to $P_3, C_3, C_4, C_5$. Then in this case $|D^d| = p - 1$.

In Theorem 2, the upper bound for $\gamma_{ddl}(G)$ shall be expressed in terms of $\gamma(G)$ and vertices of $G$.

**Theorem 2**: For any connected $(p, q)$ graph $G$, $\gamma_{ddl}(G) + diam(G) \leq p + \gamma(G)$.

**Proof**: Let $I = \{e_{x_1}e_{x_2}, e_{x_3}, ..., e_{x_n}\}$ subset of $E(G)$ be the minimal set of edges which constitutes the longest path between any two distinct vertices $u, v \in V(G)$ such that $d_{st}(u, v) = diam(G)$. Furthermore let $D = \{v_1, v_2, ..., v_k\}$ be any minimal dominating set of $G$ and let $E = \{e_{x_1}, e_{x_2}, ..., e_{x_n}\}$ be the set of edges of $G$.

Now by definition of $L(G), E(G) = V[L(G)]$. Let $D^d = \{u_1, u_2, ..., u_k\}$ be the double dominating set of $L(G)$ such that $|N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d$. It follows that $|D^d| = d_{st}(u, v) \leq p \cup |D|$ and hence $\gamma_{ddl}(G) + diam(G) \leq p + \gamma(G)$.
Theorem 3: For any connected \( (p, q) \) graph \( G, \gamma_{dd}(G) \leq q \).

Proof: Suppose \( H = \{u_1, u_2, \ldots, u_n\} \) be the subset of \( V[L(G)] \) and \( \deg(u_i) \forall u_i \in H \) has at least two. Then \( D_1 \) is subset of \( H \) form a minimal dominating set of \( L(G) \). Further if \( I = \{u_1, u_2, \ldots, u_n\} \) be the set all end vertices in \( L(G) \), then \( I \cup H = D \), where \( H \subseteq H \) form a double dominating set of \( L(G) \) such that \( |N[u] \cap D| \geq 2 \forall u \in V[L(G)] - D \). Since \( V[L(G)] = E(G) = q \), it follows that \( |D| \leq q \). Hence \( \gamma_{dd}(G) \leq q \).

Theorem 4: For any connected \( (p, q) \) graph \( G, \gamma_{dd}(G) + \gamma[L(G)] \leq p + 2 \).

Proof: Let \( D \) be the minimal dominating set of \( G \). Now in \( L(G) \), if \( F = \{u_1, u_2, \ldots, u_n\} \) be the set of all end vertices in \( L(G) \), then \( F \cup H = D \), where \( H \subseteq V[L(G)] - F \). F forms a double dominating set of \( L(G) \), such that \( |N[u] \cap D| \geq 2 \forall u \in V[L(G)] - D \). Since each vertex in \( L(G) \) corresponds to the edges of \( G \) and each edge in \( G \) is incident to two vertices of \( G \), it follows that \( |D| \leq p \). Hence \( \gamma_{dd}(G) + \gamma[L(G)] \leq p + 2 \).

Theorem 5: For any connected \( (p, q) \) graph \( G, \gamma_{dd}(G) \leq p \).

Proof: Let \( D \) be any minimal dominating set of \( G \). Further let \( E = \{e_1, e_2, \ldots, e_n\} \) be the set of all edges which are incident to the vertices of \( G \). Now by definition of line graph, \( V[L(G)] = E(G) \). Suppose \( I = \{u_1, u_2, \ldots, u_i\} \) be the set of all end vertices in \( L(G) \), then \( I \cup H = D \), where \( H \subseteq F \) subset of \( F \), forms a double dominating set of \( L(G) \) such that \( |N[u] \cap D| \geq 2 \forall u \in V[L(G)] - D \). Clearly \( |D| = |I \cup H| \leq p \). It follows that \( \gamma_{dd}(G) \leq p \).

Theorem 6: For any connected \( (p, q) \) graph \( G, \gamma_{dd}(G) + \chi(G) \leq p + \Delta(G) \).

Proof: By Theorem 1 and by Theorem 3, clearly it follows that \( \gamma_{dd}(G) + \chi(G) \leq p + \Delta(G) \).

Theorem 7: For any connected \( (p, q) \) graph \( G, \gamma_{dd}(G) + \gamma(G) \leq p + \left\lceil \frac{\Delta}{2} \right\rceil \).

Proof: Let \( B = \{v_1, v_2, \ldots, v_n\} \) be the minimum set of vertices which covers all the edges of \( G \) such that \( |B| = \alpha(G) \). Further \( D \) be a \( \gamma \)-set of \( G \). Let \( E = \{e_1, e_2, \ldots, e_n\} \) be the set of all edges of \( G \).

Now by definition of line graph \( L(G), E(G) = V[L(G)] \). Suppose \( I = \{u_1, u_2, \ldots, u_n\} \) be the set of all end vertices in \( L(G) \), then \( |I \cup H| = D \), where \( H \subseteq F \), forms a double dominating set of \( L(G) \) such that \( |N[u] \cap D| \geq 2 \forall u \in V[L(G)] - D \). It follows that \( 2|H \cup I \cup D| - |B| \leq 2p \) and hence \( \gamma_{dd}(G) + \gamma(G) \leq p + \left\lceil \frac{\Delta}{2} \right\rceil \). Suppose \( G \) is isomorphic to \( C_4 \). Then in this case, \( |B| = 2 \) and \( \alpha(G) = 2 = \alpha(G) \).

Clearly, \( \gamma_{dd}(G) + \gamma(G) \leq p + \left\lceil \frac{\Delta}{2} \right\rceil \).
Theorem 8: For any connected \((p, q)\) graph \(G\), \(\gamma_{dd}(G) \leq \gamma_5(G) + \gamma_2(G)\).

Proof: By Theorem 12 and Theorem 13 the result follows.

3. Lower Bound for \(\gamma_{dd}(G)\):

Theorem 9: For any connected \((p, q)\) graph \(G\), \(\gamma_{dd}(G) \leq \gamma_5(G)\).

Proof: Let \(D = \{v_1, v_2, \ldots, v_m\}\) be any minimal dominating set of \(G\) and let \(F = \{e_1, e_2, \ldots, e_i\}\) be the set of edges which are incident with the vertices of \(G\). Now by the definition of \(L(G)\),

\[ F \subseteq V[L(G)]. \]

Clearly \(D^d = \{u_1, u_2, \ldots, u_h\} \subseteq F\) in \(L(G)\) forms the double dominating set of \(L(G)\) such that

\[ |N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d. \]

Further, suppose \(C = \{v_1, v_2, \ldots, v_n\}\) be the set of all non end vertices in \(G\), then there exists at least one vertex \(v\) of maximum degree \(\Delta(G)\) in \(C\), such that \(|D^d|, \Delta(G) \geq p\). It follows that \(\gamma_{dd}(G) \geq \frac{p}{\Delta(G)}\).

Theorem 10: If every non end vertices of a tree is adjacent to at least one end vertices, then \(\gamma_{dd}(G) \geq p - m\). Where \(m\) is the number of end vertices in \(T\).

Proof: Let \(T^\prime\) be a tree. If \(diam(T^\prime) \geq 3\) and \(S = \{v_1, v_2, \ldots, v_m\}\) be the set of all end vertices of \(T^\prime\) with \(|S| = m\) and \(d(v_i) = 1\) \(\leq i \leq m\). Let \(E = \{e_1, e_2, \ldots, e_i\}\) be the edge set of \(T^\prime\). Now by the definition line graph \(L(G)\), \(E(G) = V[L(G)]\). Suppose \(I = \{u_1, u_2, \ldots, u_h\}\) be the set of all end vertices in \(L(G)\), then \(I \cup H = D^d\)

where \(H \subseteq E\), forms a double dominating set of \(L(G)\) such that \(|N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d\). Since for any tree \(T, q = p - 1\), it follows that \(|D^d| \geq p - |S|\) and hence, \(\gamma_{dd}(G) \geq p - m\).

Theorem 11: For any connected \((p, q)\) graph \(G, \gamma_{5}(G) \leq \gamma_{dd}(G)\).

Proof: Let \(v \in V(G)\) and \(\deg(v) = \delta(G)\). Since \(diam(G) = 2\), then by Theorem A the dominating set \(D, |D| \leq \delta(G) + 1\). Therefore, \(\gamma_{5}(G) \leq \delta(G) + 1\). Suppose for any connected graph with \(diam(G) \geq 2\), again by Theorem A, \(|D| \geq \delta(G) + 1\). Hence \(\gamma_{5}(G) \geq \delta(G) + 1\). Now let \(D^d\) be a double dominating set of \(L(G)\) such that \(|N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d\). Again by Theorem A, \(|D^d| \geq \delta[L(G)] + 2\). Clearly it follows that \(\gamma_{dd}(G) \geq \delta[L(G)] + 2\). Hence \(\gamma_{5}(G) \leq \gamma_{dd}(G)\).

Theorem 12: For any connected \((p, q)\) graph \(G, \gamma_{5}(G) \leq \gamma_{dd}(G)\).

Proof: Let \(S\) be a maximum independent set of vertices in \(G\). Then there exists a set \(S_1\) subset of \(S\) such that \(S_1\) has at least two vertices and every vertex in \(S_1\) is adjacent to some vertex in \(V - S_1\). Hence \(V - S_1\) is a split dominating set of \(G\). Therefore \(|V - S_1| \leq |S|\). Hence \(\gamma_{5}(G) \leq \beta_0\). Now let \(D^d\) be a double dominating set in \(L(G)\) such
that $|N[u] \cap D| \geq 2 \forall u \in V[L(G)] - D^d$. Since $E(G) = V[L(G)]$, and let $S^I$ be a maximum independent set of $L(G)$. Then every vertex in $S^I$ is adjacent to some vertex in $V[L(G)] - D^d$, such that $|N[v] \cap S^I| \geq 1 \forall v \in V[L(G)]$. Clearly, $|N[v] \cap S^I| \leq |N[v] \cap D^d|$ it follows that $\beta_0[L(G)] \leq \gamma_{ddl}(G)$.

Hence $\gamma_0(G) \leq \gamma_{ddl}(G)$.

**Theorem 13**: For any connected $(p, q)$ graph $G$, $\gamma_{sd}(G) \leq \gamma_{ddl}(G)$.

**Proof**: Let $\psi$ be a vertex of maximum degree $\Delta(G)$. Then $\psi$ is adjacent to $N(\psi)$ vertices such that $\Delta(G) = N(\psi)$. Hence $V - N(\psi)$ is a dominating set. Let $D$ be a connected dominating set of $G$ such that $D \leq V - \Delta(G)$. Therefore $|D| \leq |V - N(\psi)|$. Hence $\gamma_0(G) \leq p - \Delta(G)$. Now, let $D^d$ be a double dominating set of $L(G)$ such that $N[v] \cap D^d | \geq 2 \forall u \in V[L(G)] - D^d$. Also $D^d \geq V - \Delta(L(G))$. Therefore $|D^d| \geq |V - \Delta(G)|$. it follows that $\gamma_{ddl}(G) \geq p - \Delta[L(G)]$.

Hence $\gamma_0(G) \leq \gamma_{ddl}(G)$.

**Theorem 14**: For any connected $(p, q)$ graph $G$, $\gamma_{sd}(G) \leq \gamma_{ddl}(G)$.

**Proof**: Let $S$ be a maximum independent set of vertices in $G$. Then $V - S$ is a strong split dominating set of $G$.

Since $S$ is maximum, $V - S$ is minimum. Thus $\gamma_{sd}(G) = \alpha_0(G)$. Now let $D^d$ be a double dominating set in $L(G)$.

Since $E(G) = V[L(G)]$, let $S^I$ be a maximum independent set of $L(G)$. Then $V[L(G)] - S^I$ is minimum and $|V[L(G)] - S^I| \leq |D^d|$. Clearly it follows that $\alpha_0[L(G)] \leq \gamma_{ddl}(G)$. Hence $\gamma_{sd}(G) \leq \gamma_{ddl}(G)$.

**Theorem 15**: For any connected $(p, q)$ graph $G$, $\gamma_{ns}(G) \leq \gamma_{ddl}(G)$.

**Proof**: By Theorem[4], a non-split dominating set $D$ of $G$ is minimal if and only if for each vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$. Therefore $|N(u) \cap D| = 1$. Now let $D^d$ be a double dominating set of $L(G)$ such that $|N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d$. From the above, if for each vertex $v \in D^d$ then there exists a vertex $u \in V - D^d$ such that $N(u) \cap D = \{v_i, v_j\}$ for $i \neq j$ and $1 \leq i, j \leq m$. Therefore $|N(u) \cap D| = 2$. It is clear that $|N(u) \cap D| \leq |N(u) \cap D^d|$. Hence $\gamma_{ns}(G) \leq \gamma_{ddl}(G)$.

**Theorem 16**: For any connected $(p, q)$ graph $G$, $\gamma_{ns}(G) \leq \gamma_{ddl}(G)$.

**Proof**: Let $E = \{e_{i_1}, e_{i_2}, ..., e_{i_n}\}$ be the set of edges of $G$. Let $D = \{v_1, v_2, ..., v_k\}$ be any minimal dominating set of $G$ such that for every vertex $u \in V(G) - D$ such that $|N[u] \cap D| \geq 1$. Now by definition of $L(G)$, $V[L(G)] = E(G)$, let $D^d = \{u_1, u_2, ..., u_n\}$, $1 \leq l \leq n$, in $L(G)$, forms the double dominating set of $L(G)$, such that $|N[u] \cap D^d| \geq 2 \forall u \in V[L(G)] - D^d$. It follows that $|D| \leq |D^d|$ and

www.tjprc.org
hence \( y(G) \leq y_{da}(G) \).

**Theorem 17:** For any connected \((p, q)\) graph \(G, \left\lfloor \frac{\text{diam}(G)}{3} \right\rfloor \leq y_{da}(G)\).

**Proof:** By Theorem [8] and Theorem [16] the result follows.

**REFERENCES**

2. F. Harary, “Graph Theory”, Adison Wesley, Reading Mass (1972)