ON THE LICT GEODETIC NUMBER OF A GRAPH

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ABSTRACT
For any graph $G=(V,E)$, the lict graph of $G$ denoted by $G_\eta$. The Lict graph $G_\eta$ of a graph $G$ as the graph whose vertex set is the union of the set of edges and the set of cut vertices of $G$ in which two vertices are adjacent if and only if the corresponding edges of $G$ are adjacent or the corresponding members are incident. For two vertices $u$ and $v$ of $G$, the set $I(u,v)$ consists of all vertices lying on a $u-v$ geodesic in $G$. If $S$ is a set of vertices of $G$, then $I(S)$ is the union of all sets $I(u,v)$ for vertices $u$ and $v$ in $S$. The geodetic number $g(G)$ is the minimum cardinality among the subsets $S$ of $V(G)$ with $I(S)=V(G)$. In this paper we obtain the geodetic number of lict graph of any graph. Also, obtain many bounds on geodetic number in terms of elements of $G$.

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INTRODUCTION
For any graph $G=(V,E)$, the Lict graph $G_\eta$ whose vertex set is the union of the set of edges and the set of cut vertices of $G$ in which two vertices of $G_\eta$ adjacent if and only if the corresponding edges of $G$ are adjacent or the corresponding members of $G$ are incident.

The distance $d(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, rad $G$, and the maximum eccentricity is its diameter, diam $G$. A $u-v$ path of length $d(u,v)$ is called a $u-v$ geodesic. We define $I(u,v)$ to be the set (interval) of all vertices lying on some $u-v$ geodesic of $G$, and for a nonempty subset $S$ of $V(G)$,

$$I(S) = \bigcup_{u,v \in S} I(u,v).$$

A set $S$ of vertices of $G$ is called a geodetic set in $G$ if $I(S)=V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in $G$ is called the geodetic number $g(G)$.
Now we define geodetic number of lict graph of a graph $G$. A set $S'$ of vertices of $\eta(G) = H$ is called a geodetic set in $H$ if $I(S') = V(H)$, and a geodetic set of minimum cardinality is a lict geodetic number of $G$ and is denoted by $g_{\eta}(G)$.

The Cartesian product of two graphs $G$, $H$ by $G \times H$, and it is the graph with the vertex set $V(G) \times V(H)$ specified by putting $(u,v)$ adjacent to $(u',v')$ if and only if (1) $u = u'$ and $v \in E(H)$ or

(2) $v = v'$ and $uu' \in E(G)$. A vertex $v$ is an extreme vertex in a graph $G$, if the sub graph induced by its neighbors is complete. A vertex cover in a graph $G$ is a set of vertices that covers all edges of $G$. The minimum number of vertices in a vertex cover of $G$ is the vertex covering number $\alpha(G)$ of $G$. An edge cover of a graph $G$ without isolated vertices is a set of edges of $G$ that covers all the vertices of $G$. The edge covering number $\alpha(G)$ of a graph $G$ is the minimum cardinality of an edge cover of $G$. A set of vertices in $G$ is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of $G$ and is denoted by $\beta(G)$. An independent set of edges of $G$ has no two of its edges are adjacent and the maximum cardinality of such a set is the edge independence number $\gamma(G)$.

For any undefined term in this paper, see [5, 6].

PRELIMINARIES

We need the following theorems for our further results

Theorem 2.1[6]. For any graph $G$ of order $n$, $\alpha(G) + \beta(G) = n$.

Theorem 2.2[3]. For a complete bipartite graph $K_{r,s}$, the edge covering number is $\alpha(K_{r,s}) = s$ if $1 \leq s \leq r$.

Theorem 2.3[5]. For any path $P_n$, the edge covering number is $\alpha(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

Theorem 2.4[4]. Every geodetic set of a graph contains its extreme vertices.

Theorem 2.5[4]. If $G$ is a nontrivial connected graph, then $g(G) \leq g(G \times K_2)$.

We start with following propositions.

Proposition 1. The pendant edges and cut vertices of a tree are the extreme vertices of a lict graph $\eta(G)$.

Proposition 2. Geodetic number of $K_{1,1}, K_{1,2}, ..., K_{1,n}$ is the number of non cut vertices of $G$.

Proposition 3. For any path $P_n$, $\eta(P_n) = K_3 \cdot K_3 \cdot (n-2)$ factors.
MAIN RESULTS

**Theorem 3.1.** For any tree $T$ with $n$ vertices, $g_\eta(T) = n$.

**Proof.** Let $S$ be the set of all extreme vertices of a lict graph of a tree $T$. By the Theorem 2.5, $g_\eta(T) \geq S$. On the other hand, for an internal vertex $v$ of $T$, there exits end vertices $x, y$ of $T$ such that $v$ lies on the unique $x-y$ geodesic in $T$. Thus $v \in I(S)$ and $I(S) = V(T)$, then $g_\eta(T) \leq S$. Therefore, $g_\eta(T) = S$.

Also, every geodetic set $S'$ of $T$ must contain $S$ is the unique minimum geodetic set. By the proposition 1, the pendant edges and cut vertices of a tree are the extreme vertices of a lict graph. Since, the number of vertices in tree $T$ is the sum of cut vertices and number of pendent edges and hence the minimum geodetic set is the number of vertices $n$ in tree $T$. Thus $g_\eta(T) = n$.

**Corollary 3.1.1.** For any path $P_n$ with $n$ vertices, the lict geodetic number $g_\eta(P_n) = n$.

**Proof.** Proof follows from Theorem 3.1.

**Theorem 3.2.** For any tree $T$ the lict geodetic number is $g_\eta(T) = \alpha_1(T) + \beta_1(T)$.

**Proof.** By the Theorem 2.1, $\alpha_1(T) + \beta_1(T) = n$. Since $g_\eta(T) = n$, we have $g_\eta(T) = \alpha_1(T) + \beta_1(T)$.

**Theorem 3.3.** For any star $K_{1,r}$, the geodetic number $g_\eta(K_{1,r}) = \alpha_1(K_{1,r}) + 1$.

**Proof.** Let $e_1, e_2, e_3, \ldots, e_r$ be the pendant edges and $c_1$ be the cut vertex of $K_{1,r}$. Let $S$ be the set of all extreme vertices of the lict graph of $K_{1,r}$. Therefore $g_\eta(K_{1,r}) = S$. By the definition of lict graph, $\eta(K_{1,r}) = K_{1,r+1}$. Also by the proposition 1, the pendant edges and the cut vertices of a $K_{1,r}$ are the extreme vertices of a lict graph. Since $\alpha_1(K_{1,r}) = s$, we have the geodetic number of lict graph of $K_{1,r}$ is the sum of edge covering number and the cut vertex. Hence $g_\eta(K_{1,r}) = \alpha_1(K_{1,r}) + 1$.

**Theorem 3.4.** For any path $P_n$ with $n$ vertices, the lict geodetic number $g_\eta(P_n) = L(P_n) + c_i$, where $L(P_n)$ is line graph of $P_n$ and $c_i$ be the number of cutvertices.

**Proof.** Let $P_n$ be the path with $n \geq 3$ vertices. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices and let $e = \{v_i, v_{i+1}, i = 1, 2, 3, \ldots, n\}$ be the edge set of path $P_n$. By the definition of line graph, $L(P_n) = P_{n-1}$ and $g(P_n) = 2$. Since $L(P_n)$ is a sub graph of $g_\eta(P_n)$, by the definition of lict graph, cut vertices $c_i$ are adjacent to the vertices $e_1, e_2, e_3, \ldots, e_i$ of $L(P_n)$. Hence $g_\eta(P_n) = L(P_n) + c_i$.

**Theorem 3.5.** For any path $P_n$ with $n$ vertices, the geodetic number

$$g_\eta(P_n) = \begin{cases} 2\alpha_1(P_n) & \text{if } n \text{ is even} \\ 2\alpha_1(P_n) & \text{if } n \text{ is odd} \end{cases}$$

where $\alpha_1(P_n)$ is an edge covering number.

**Proof.** Let $P_n$ be the path with $n \geq 3$ vertices. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices and let $e = \{v_i, v_{i+1}, i = 1, 2, 3, \ldots, n\}$ be the edge set of the path $P_n$. By the Theorem 2.3, the edge covering number $\alpha_1(G)$ of a graph is a minimum cardinality of an edge cover of $G$.

**Case (1).** Suppose $n$ is even, by the Theorem 2.3, $\alpha_1(P_n) = \frac{n}{2}$

$$\Rightarrow n = 2\alpha_1(P_n)$$

Since $g_\eta(P_n) = n$, we have $g_\eta(P_n) = 2\alpha_1(P_n).$
Case (2). Suppose $n$ is odd, by the Theorem 2.3, $\alpha_1(P_n) = \frac{n+1}{2}$, 
\[ \Rightarrow n+1 = 2 \alpha_1(P_n) \]
\[ \Rightarrow n = 2 \alpha_1(P_n) - 1 \]
Since $g_\eta(P_n) = n$, we have $g_\eta(P_n) = 2 \alpha_1(P_n) - 1$. Hence the result.

**Theorem 3.6.** For any path $P_n$ with $n$ vertices, the lict geodetic number of $K_2 \times G$ is $g_\eta(K_2 \times \eta(P_n)) = n$.

**Proof.** By the definition of lict graph, $\eta(P_n) = K_3 \cdot K_3 \cdot \ldots (n-2)$ factors for $n \geq 3$. Consider $G = \eta(P_n)$. Let $K_2 \times G$ be graph formed from two copies $G_1$ and $G_2$ of $G$ and $S$ be a minimum geodetic set of $K_2 \times G$. Now, we define $S'$ to be the union of $\{x\}$ belonging to $S$ and the vertices of $G_1$ corresponding to vertices of $G_2$ belonging to $S$. Let $G = C_n$ be the cycle with $n$ vertices, the lict geodetic number of $K_2 \times G$ is $g_\eta(K_2 \times \eta(P_n)) = n$.

Conversely, let $S$ contains a vertex $x$ with the property that every vertex of $G_i$ lies on an $x$-$w$ geodesic for some $w \in S$. Let $S'$ consists of $x$ together with those vertices of $G_2$ corresponding to those $S\{-x\}$. Thus $|S'| = |S|$. We show that $S'$ is a geodetic set of $K_2 \times G$. Hence $g(K_2 \times G) \leq g(G)$. Thus $g_\eta(K_2 \times \eta(P_n)) = n$.

**Corollary 3.6.1.** For any path $P_n$ with $n$ vertices, the lict geodetic number $g_\eta(K_3 \times \eta(P_n)) = n$.

**Proof.** Proof follows from Theorem 3.6.

**Theorem 3.7.** Let $G'$ be the graph obtained by adding a pendant edge $\{u, v\}$ to a cycle $G = C_n$ with $u \in G$ and $v \in G$, then $g_\eta(G') = 3$.

**Proof.** Let $\{e_1, e_2, e_3, \ldots, e_n, e_l\}$ be a cycle with $n$ vertices which is odd and let $G'$ be the graph obtained from $G = C_n$ by adding a pendant edge $\{u, v\}$ such that $v \notin G$, $u \in G$. By the definition of lict graph, $\eta(G')$ has $K_4$ as an induced sub graph, also the edge $\{u, v\} = e_1$ becomes a vertex of $\eta(G')$. Further, the cut vertex $u$ is also a vertex of $\eta(G')$ and it belongs to some geodetic set of $\eta(G')$. Let $e_i = \{a, b\} \in G$ such that $d(u, a) = d(u, b)$ in the graph $\eta(G)$, 2 elements subset of $S$ of $\eta(G')$ has the property that $I(S) = V[\eta(G')]$. Thus $g_\eta(G') \geq 3$. Since $S = \{a, e_i, e_j\}$ is a geodetic set, we have $g_\eta(G') = 3$.

**Theorem 3.8.** Let $G'$ be the graph obtained by adding a pendant edge $\{u, v\}$ to a cycle $C_n$ with $u \in G$ and $v \in G$, then $g_\eta(G') = g(G') + g(G) = 4$.

**Proof.** Let $\{e_1, e_2, e_3, \ldots, e_n, e_l\}$ be a cycle with $n$ vertices which is even and let $G'$ be the graph obtained from $G = C_n$ by adding a pendant edge $\{u, v\}$ such that $v \notin G$, $u \in G$. By the definition of lict graph, $\eta(G')$ has $K_4$ as an induced sub graph, also the edge $\{u, v\} = e_1$ becomes a vertex of $\eta(G')$. Further, the cut vertex $u$ is also a vertex of $\eta(G')$. Hence $\{u, e_i, e_j\}$ are the vertices of $\eta(G')$ where $e_i, e_j$ are the edges incident on the antipodal vertex of $u$ in $G'$ and these vertices belongs to some geodetic set of $\eta(G')$. Thus, $\eta(G') = C_n \cup K_4$. Clearly $g(G') = 2$ and $g(G) = 2$. Hence $g_\eta(G') = g(G') + g(G) = 4$. 

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Theorem 3.9. Let $G'$ be the graph obtained by adding a pendant edge $u_{i}v_{j}$, $i=1,2,3,...,n$, $j=2,3,...,k$ to each vertex of $G$ such that $v_{j} \notin G, u_{i} \in G$ then $g_{\eta}(G') = g(G') + k$ where $k$ be the number of pendant vertices of $G$.

Proof. Let $\{e_{1},e_{2},...,e_{n},e_{1}\}$ be a cycle and $G=C_{n}$ with $n$ vertices. Let $G'$ be the graph obtained by adding a pendant edge $\{u_{i},v_{j}\}$, $i=12,3,4,...,n$, $j=2,3,...,k$ to each vertex of $G$ such that $v_{j} \notin G$ and $u_{i} \in G$. Let $k$ be the number of pendant vertices of $G'$ and $u_{i}, i=1,2,3,...,n$ be the cut vertices in $G'$. By the definition of lict graph, $\eta(G')$ have $k$ copies of $K_{4}$ as an induced sub graph. If $(u_{i}, v_{j}) = e_{i}$ then $e_{1},e_{2},...,e_{k}$ and $u_{i}$ becomes the vertices of $\eta(G')$ and these elements belongs to some geodetic set of $\eta(G')$ say $g_{\eta}(G')$.

Now let $S'$ be the geodominating set of $G'$ containing $u_{i}, v_{j}$ and the $e_{i}$. Clearly these are the extreme vertices of $\eta(G')$. By the Theorem 3.1, $g_{\eta}(G') = g(G') + k$.

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